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MODEL BUILDING

IN F-THEORY

USING HYPERCHARGE FLUXES

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MODELLBILDUNG IN F-THEORIE MITTELS HYPERLADUNGSFLÜSSEN:

Eine der Herausforderungen der String-Phänomenologie ist es, Kompaktifizierungen der Superstringtheorie zu finden sodass die resultierende effektive Niedrigenergietheorie dem Minimalen Supersymmetrischen Standardmodell der Teilchenphysik ähnelt, d.h. mit $SU(3) \times SU(2) \times U(1)$ Eichgruppe, drei Generationen von Fermionen und einem Higgs-Doublet. Eine etablierte Methode um die korrekte Eichgruppe zu erreichen ist, eine SU(5) GUT-Theorie mit D7-Branen in Typ IIB Superstringtheorie zu modellieren, und dann die GUT-Gruppe mittels eines Hyperladungsflusses zu brechen.

Die korrekte nicht-störungstheoretische Beschreibung von Typ IIB Superstringtheorie mit D3- und D7-Branen ist F-Theorie. In F-Theorie wird die Geometrie der Membranen und der Kompaktifizierung durch eine vierdimensionale elliptisch gefaserte Calabi-Yau Mannigfaltigkeit erfasst. In [1] geben die Autoren ein einfaches Beispiel für ein realistisches Modell in diesem Zusammenhang, welches jedoch neun Generationen von Fermionen und fünf Higgs-Doublets beinhaltet.

Das Ziel dieser Arbeit ist es, die notwendigen Voraussetzungen in der Differentialgeometrie aufzuarbeiten, um im Detail zu erklären, wie ein solches Modell in F-Theorie aufgebaut werden kann, und schließlich das Modell aus [1] zu verbessern. Durch das Hinzufügen eines sogenannten G_4^λ-Flusses konnten wir die Anzahl der Generationen in dem Modell auf den gewünschten Wert von drei zu reduzieren.

MODEL BUILDING IN F-THEORY USING HYPERCHARGE FLUXES:

One of the challenges in string phenomenology is to find compactifications of superstring theory such that the resulting low-energy effective theory resembles the Minimal Supersymmetric Standard Model of particle physics with its $SU(3) \times SU(2) \times U(1)$ gauge group, three generations of fermions and one Higgs doublet. An established method that achieves the correct gauge group is to model an SU(5) GUT theory using type IIB superstring theory with D7-branes and then to break the GUT group by including a hypercharge flux.

The correct non-perturbative description of type IIB superstring theory with D3- and D7-branes is F-theory. In F-theory, the geometry of the branes and of the compactification space is captured by a four-dimensional elliptically fibered Calabi-Yau. In [1], the authors give a basic example of a realistic model in this context, including however nine generations of fermions and five Higgs doublets.

The aim of this thesis is to review the necessary prerequisites in differential geometry, to explain in detail how such a model in F-theory can be built, and finally to improve the model given in [1]. By including a so-called G_4^λ-flux, we were able to reduce the number of generations in the model to the desired value of three.

Contents

Chapter 1

Introduction

It has always been a goal of Physics to fundamentally understand the laws of the universe and find one single model which is valid for all kinds of processes in it. Unification is a recurring theme along the way, starting with Maxwell in the 19th century: He was able to describe the hitherto seperate phenomena of electricity and magnetism in a unified framework, it is now called electromagnetism. After further unifications, we are nowadays describing the electromagnetic, the weak and the strong interactions between all known elementary particles in just one model, the Standard Model of particle physics.

Still, we know that the Standard Model can not be valid for arbitrarily high energy scales, and it is believed that it has to be unified with gravity in order to find a fundamental theory that is UV-complete, i.e. valid on all scales. The best developed and arguably most promising candidate for this unification is superstring theory. Superstring theory is a ten-dimensional quantum theory that necessarily predicts gravity. The theory is unique in ten dimensions, but needs to be compactified on a six-dimensional manifold in order to connect to the real world. The field of string phenomenology is concerned with how the effective particle spectrum in the low-energy limit depends on the geometry of this compactification. One of the main challenges is to find compactifications such that the low-energy effective theory resembles the Standard Model. As superstring theory is inherently supersymmetric, it is easier to aim instead for a supersymmetric version of the Standard Model such as the Minimal Supersymmetric Standard Model (MSSM).

The MSSM is a non-abelian gauge theory with an $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge group, three generations of chiral matter and one Higgs doublet. There is an established method that gives us the correct gauge group from string compactification, it was first discovered in [2] (see also [3]). The idea is to first model an SU(5) grand unified theory using certain 8-dimensional non-perturbative objects called D7-branes. If one then includes a hypercharge flux, a non-zero background value for the field strength corresponding to the hypercharge generator embedded in SU(5), the symmetry is broken down to the Standard Model gauge group or a subgroup of it. To make sure that the full Standard Model symmetry is retained, one has to take care that the hypercharge generator does not acquire a Stückelberg mass. This sets a condition on the hypercharge flux, the best language to formulate this condition is in the homology of the compactification manifold X_3: The stack of D7-branes lies on a divisor $S \subset X_3$ and the hypercharge flux is a 2-form $F^Y \in H^2(S)$. F^Y is Poincaré dual to a 2-cycle $\mathcal{C}_Y \in H_2(S)$ which needs to be non-trivial in the homology of S but trivial in the homology of X_3,

$$\iota_* \mathcal{C}_Y = 0 \tag{1.1}$$

(where $\iota : S \to X_3$ is the embedding).

In the presence of D7-branes, the string coupling varies highly as a function of spacetime. The correct non-perturbative description of type IIB string compactifications with D3- and D7-branes is F-theory. The F-theory conjecture [4] states that the physics of the type IIB orientifold is encoded in the geometry of a four-dimensional elliptically fibered Calabi-Yau Y_4 [5]. The location of the branes is encoded in the singularities of the elliptic fibration, the type IIB brane fluxes and closed string fluxes can be described as components of a 4-form flux $G_4 \in H^4(Y_4)$. The construction described above which yields the MSSM gauge group can be lifted to F-theory, where the F^Y-flux corresponds to a G_4^Y-flux (see e.g. [6–8]). By additionally including so-called G_4^X- and G_4^λ-fluxes, we can control the amount of chiral matter in the low-energy effective theory [9].

In [1], the authors give a basic example of a realistic model in F-theory where the MSSM gauge group is achieved in the manner discussed above. They utilize complete intersections in toric varieties to describe a hypercharge flux satisfying the condition (1.1) and then also turn on G_4^X-flux to achieve doublet-triplet splitting of the SU(5) Higgs vectors. Being able to implement doublet-triplet splitting is a huge advantage of this way of breaking the SU(5) GUT compared to other non-stringy approaches. The model described in [1] has one drawback however, as it includes nine generations of chiral matter and five Higgs doublets.

The aim of this thesis is to first review the necessary prerequisites in differential geometry, to explain in detail how that model can be constructed in F-theory, and then to improve it.

Overview. In chapter 2, we will give a very brief overview over the most important mathematical concepts and formulas that we will need in the rest of the thesis. They mainly belong to the fields of differential and algebraic geometry: We explain how to work with complex manifolds, focusing on fibrations over toric varieties and complete intersections within those. Especially, we will have to make calculations in homology and for example calculate Chern classes of such manifolds. Further, we need to address the theory of the Lie group SU(5) and its representations, and how the Standard Model is embedded in this group. The mathematical appendix, chapters A through C, is meant to serve as a much more detailed review of these topics.

We go on in chapter 3 by reviewing the motivation for considering GUT theories in particle physics and explaining in detail how an SU(5) GUT theory can be broken to the Standard Model. The next step is to introduce type IIB superstring theory, discussing especially the low-energy effective action of the strings and D-branes in the theory.

In chapter 4, we realize that this description is a bit inelegant and that there is a simpler description in terms of an elliptically fibered Calabi-Yau 4-fold, thanks to an $\mathrm{SL}(2,\mathbb{Z})$-symmetry of type IIB theory. This non-perturbative description is F-theory, we will mainly be interested in how the singularity structure of the 4-fold corresponds to branes in the viewpoint of type IIB theory. Having understood this connection, we can use F-theory to model a generic SU(5) GUT theory.

In chapter 5 we will introduce the symmetry breaking mechanism mentioned above. First we will understand how Stückelberg masses arise in the context of string theory and then see why turning on a hypercharge flux satisfying condition (1.1) generates a mass for precisely those gauge bosons in an SU(5) GUT such that it will be broken to the Standard Model. Afterwards, we can lift that type IIB hypercharge flux F^Y to the corresponding object in F-theory, G_4^Y-flux. We also give a first basic example of a model with an SU(2) gauge group instead of SU(5) to familiarize ourselves with the kind of calculations appearing later.

Finally, we'll fully explain the construction of the SU(5) model outlined in [1] in chapter 6. We begin with defining the compactification manifold X_3, the brane locus S and a flux F^Y with (1.1). Also, we'll review how matter is encoded in matter surfaces in the elliptic fibration and how to use G-flux to count the amount of matter in the effective theory. By constraining the geometry of the elliptic fibration in a clever way, the Higgs surface splits and doublet-triplet splitting can be achieved if G_4^X-flux is included as well. This leads to the final result from [1] with nine generations and five Higgs doublets. Section 6.6 discusses our attempts to improve on this by additionally including G_4^λ-flux. In particular, we will show that we can reduce the number of generations in the model to the desired value of three.

Chapter 7 summarizes the construction detailed in chapter 6 and the resulting spectrum.

Chapter 2

Mathematical Foundations

This chapter introduces the most important mathematical results that will be needed for the rest of the work. In the following sections 2.1 and 2.2, we will mainly summarize the results of the Mathematical Appendix, chapters A through C. Then, in section 2.3, we will include some facts about the theory of the Lie groups $\mathrm{SU}(N)$. Those will be needed later when we introduce Grand Unified Theories.

2.1 Differential Geometry

2.1.1 Real and Complex Manifolds

Differentiable Manifolds. We will assume that the reader is familiar with the basic definition of a differentiable manifold M of dimension m and its (co-)tangent spaces T_pM and T_p^*M for $p \in M$. An (r, s)-tensor is an element of $T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}$ and a tensor field is the smooth assignment of a tensor to each point of the manifold. The set of smooth functions from M to $\mathbb{R}$ will be denoted $\Omega^0(M) = C^\infty(M)$, vector fields $\mathrm{Vect}(M)$ are just $(1, 0)$-tensor fields, and $(0, 1)$-tensor fields are called one-forms $\Omega^1(M)$. The differential of a function f is the one-form

$$\mathrm{d}f = \partial_\mu f \ \mathrm{d}x^\mu \ . \tag{2.1}$$

Induced Maps. Consider a smooth map $\Phi : M \to N$ between two manifolds. Φ induces a number of pullback- and pushforward-maps:

- The pullback of $f \in C^\infty(N)$ is $\Phi^* f \in C^\infty(M)$ given by $\Phi^* f = f \circ \Phi$.
- Tangent vectors $v \in T_pM$ can be pushed forward to $\Phi_* v$ and cotangent vectors $\omega \in T^*_{\Phi(p)}N$ can be pulled back to $\Phi^*\omega$. $\Phi_* v$ and $\Phi^*\omega$ are defined by writing down how they act on functions and vectors, respectively (see subsection A.1.4).
- And finally, the pullback of a one-form $\omega \in \Omega^1(N)$ is $(\Phi^*\omega)\big|_p = \Phi^* \left(\omega|_{\Phi(p)} \right)$.

The differential is compatible with the pullback that we have introduced: Let $f \in C^\infty(N)$ be a function, then

$$\Phi^*(\mathrm{d}f) = \mathrm{d}(\Phi^* f) \ . \tag{2.2}$$

Metrics. A metric g is a symmetric $(0, 2)$-tensor field that is positive definite (Riemannian metric) or at least non-degenerate (pseudo-Riemannian metric). The inverse metric, also denoted by g, is a $(2, 0)$-tensor field such that $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$.

The most important metric for us will be the *Minkowski metric*

$$g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1) \tag{2.3}$$

of flat spacetime. Its signature is $s = 3$ because three of the eigenvalues are positive.

On an orientable manifold, a metric allows us to define a canonical volume form:

$$\mathrm{vol} = \sqrt{|g|}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \; . \tag{2.4}$$

Complex Manifolds. A complex manifold X of dimension m is defined completely analogous to a differential manifold, but the charts now are maps to $\mathbb{C}^m$ and the transition functions are not only smooth, but holomorphic. The sheaf of holomorphic functions on X is called the structure sheaf $\mathcal{O}_X$.

The most important example is *complex projective space* $\mathbb{P}^m$, it can be described by $m+1$ homogeneous coordinates $[z_0 : \cdots : z_m] \neq [0 : \cdots : 0]$ with the identification

$$[z_0 : \cdots : z_m] = [\Lambda z_0 : \cdots : \Lambda z_m] \; . \tag{2.5}$$

A second example is the *torus* T^2. Two points $\omega_1, \omega_2 \in \mathbb{C}$ with $\Im(\omega_2/\omega_1) > 0$ define a lattice $L = \{n_1\omega_1 + n_2\omega_2 : n_i \in \mathbb{Z}\}$. The quotient $T^2 = \mathbb{C}/L$ is a complex torus. An important fact is that two lattices L and $\tilde{L}$ define the same torus if and only if the moduli $\tau = \frac{\omega_2}{\omega_1}$ are related by a $\mathrm{PSL}(2,\mathbb{Z})$ transformation

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d} \tag{2.6}$$

($a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$).

Kähler Manifolds. First of all, we define tangent- and cotangent spaces of a complex manifold just like for a differential manifold. After complexifying those ($T_pX^{\mathbb{C}} = \mathbb{C} \otimes T_pX$), we can write a basis in the form of the vectors ∂_μ and $\bar{\partial}_\mu$.

A metric of the form $g = g_{\mu\bar{\nu}}\mathrm{d}z^\mu \otimes \mathrm{d}\bar{z}^\nu + g_{\bar{\mu}\nu}\mathrm{d}\bar{z}^\mu \otimes \mathrm{d}z^\nu$ is called a hermitian metric, and

$$\Omega = \mathrm{i}g_{\mu\bar{\nu}}\,\mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu \tag{2.7}$$

is its Kähler form. The manifold is a Kähler manifold if $\mathrm{d}\Omega = 0$. This is an important definition because Riemann curvature tensor of a Kähler manifold enjoys an additional symmetry, which later helps to characterize Calabi-Yau manifolds.

As an example, all projective manifolds (i.e. submanifolds of complex projective space) are compact Kähler manifolds.

2.1.2 Homology and Cohomology

Homology Groups. A real r-simplex is the convex hull of $r + 1$ affinely independent points in a real vector space. A singular r-simplex in a differentiable manifold M is the image of a real r-simplex σ_r under a smooth map $f : \sigma_r \to M$. We are interested in $C_r(M)$, the free $\mathbb{R}$-module over the set of singular r-simplexes in M. Elements of $C_r(M)$ are called singular r-chains.

If M and N are differentiable manifolds and $\Phi : M \to N$ is smooth, a chain $c = f(\sigma_r) \in C_r(M)$ can be pushed forward via Φ: The pushforward Φ_*c is an element of $C_r(N)$,

$$\Phi_*c = (\Phi \circ f)(\sigma_r) = \Phi(c) \; . \tag{2.8}$$

Next, we'll consider the boundary operator $\partial : C_r(M) \to C_{r-1}(M)$, which maps a singular simplex to the (oriented) sum of its faces, those faces are $(r-1)$-simplexes. The chains $C_r(M)$ together with the boundary operators ∂ form a chain complex, meaning that $\partial^2 = 0$. For every chain complex, we can define homology groups via the following procedure:

- The group of *r-cycles* $Z_r(M)$ is the kernel of $\partial : C_r(M) \to C_{r-1}(M)$.
- The group of *r-boundaries* $B_r(M)$ is the image of $\partial : C_{r+1}(M) \to C_r(M)$.
- The r-th *singular homology group* is

  $$H_r(M) = Z_r(M)/B_r(M) \; , \tag{2.9}$$

  its dimension is the r-th Betti number $b_r(M)$.

The pushforward $\Phi_*[c]$ of a chain class $[c]$ is defined to be the class $[\Phi_*c]$.

Differential Forms. In order to define cohomology groups, we must first introduce differential forms. For an arbitrary vector space V with basis $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$, the space $\Lambda^q V$ of q-vectors is spanned by $\binom{n}{q}$ basis elements $\bigwedge_{k=1}^q \boldsymbol{e}_{i_k}$ $(1 \leq i_1 < \cdots < i_q \leq n)$, where $\bigwedge$ is the antisymmetrization of the tensor product. We choose the normalization of the components of such a q-vector ω such that

$$\omega = \omega_{i_1 \dots i_q} \bigotimes_{k=1}^{q} \boldsymbol{e}_{i_k} = \frac{1}{q!} \omega_{i_1 \dots i_q} \bigwedge_{k=1}^{q} \boldsymbol{e}_{i_k} \; . \tag{2.10}$$

A differential form now is a smooth assignment of an element of $\Lambda^q T_p^* M$ to each point p in a manifold M, the space of q-forms is written $\Omega^q(M)$.

We will list the definitions of some basic operations on differential forms:

- The *exterior product* of a q-form ω and an r-form η has components

$$(\omega \wedge \eta)_{i_1 \dots i_{q+r}} = \frac{1}{q!r!} \omega_{[i_1 \dots i_q} \eta_{i_{q+1} \dots i_{q+r}]} \; . \tag{2.11}$$

  It has the important property $\omega \wedge \eta = (-1)^{qr} \, \eta \wedge \omega$.

- The *exterior derivative* of a q-form ω is

$$\mathrm{d}\omega = \frac{1}{q!} \left(\partial_\nu \omega_{\mu_1 \dots \mu_q}\right) \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_q} \; . \tag{2.12}$$

  Also this operation is compatible with pullback, $\Phi^*(\mathrm{d}\omega) = \mathrm{d}(\Phi^*\omega)$.

- A differential form ω can be integrated over a chain c, $\int_c \omega = \int_{\sigma_q} f^*\omega$. Stokes' theorem holds:

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega \; . \tag{2.13}$$

- If a metric is given, there is a natural inner product $\langle \omega, \eta \rangle = \frac{1}{q!} \omega^{\mu_1 \cdots \mu_q} \eta_{\nu_1 \dots \nu_q}$ of two q-forms ω and η. This induces a canonical isomorphism between $\Omega^q(M)$ and $\Omega^{m-q}(M)$, the *Hodge star operator* with

$$\eta \wedge * \omega = \langle \eta, \omega \rangle \; \mathrm{vol} \; , \tag{2.14}$$

  that means

$$(*\omega)_{\nu_1 \dots \nu_{m-q}} = \frac{\sqrt{|g|}}{q!} \, \omega_{\mu_1 \dots \mu_q} \epsilon^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_{m-q}} \; . \tag{2.15}$$

  for its components. Note that $*^2 = (-1)^{q(m-q)+s}$ for a manifold of signature s.

Cohomology Groups. Since $\mathrm{d}^2 = 0$, the differential forms $\Omega^q(M)$ together with d form a cochain complex. In analogy to the homology groups, we define the *cohomology groups*

$$H^q(M) = Z^q(M)/B^q(M) \; . \tag{2.16}$$

The pullback $\Phi^*[\omega]$ of a cohomology class ω is defined to be the class $[\Phi^*\omega]$.

There are two important dualities between the homology and cohomology groups: Firstly, the cohomology group $H^q(M)$ is the dual of $H_q(M)$, this is called de Rham duality. And secondly, $H^q(M)$ is also the dual space of $H^{m-q}(M)$.

Therefore, H_q, H^q, H_{m-q} and H^{m-q} are all isomorphic, and there is a canonical isomorphism between $H_q(M)$ and $H^{m-q}(M)$. This canonical isomorphism is called *Poincaré duality*: The Poincaré dual of a class $c \in H_q(M)$ is a class $\gamma \in H^{m-q}(M)$ such that

$$\int_c \omega = \int_M \omega \wedge \gamma \quad \forall \omega \; . \tag{2.17}$$

We will often write $\gamma = [c]_M$ or just $\gamma = [c]$.

Complex Cohomology. On complex manifolds, differential forms are made up of terms

$$\omega = \frac{1}{r!s!}\omega_{\mu_1\dots\mu_r\nu_1\dots\nu_s}\,\mathrm{d}z^{\mu_1}\wedge\dots\wedge\mathrm{d}z^{\mu_r}\wedge\mathrm{d}\bar{z}^{\nu_1}\wedge\dots\wedge\mathrm{d}\bar{z}^{\nu_s}\ . \tag{2.18}$$

Such a form is called an (r,s)-form, it is an element of $\Omega^{r,s}(X)$. Obviously, $\Omega^q(X)^{\mathbb{C}} = \bigoplus_{r+s=q}\Omega^{r,s}(X)$.

Accordingly, d can be written as the sum of two operators $\partial + \bar{\partial}$. For every r,

$$0 \to \Omega^{r,0} \xrightarrow{\bar{\partial}} \Omega^{r,1} \xrightarrow{\bar{\partial}} \cdots \tag{2.19}$$

is a cochain complex with cohomology groups $H^{r,s}(X)$. Their dimensions are the *Hodge numbers* $b^{r,s}$.

The Hodge numbers are usually arranged in the Hodge diamond

$$\begin{pmatrix} & & b^{m,m} & & \\ & \iddots & \vdots & \ddots & \\ b^{m,0} & \cdots & \cdots & \cdots & b^{0,m} \\ & \ddots & \vdots & \iddots & \\ & & b^{0,0} & & \end{pmatrix}\ . \tag{2.20}$$

In a Kähler manifold, the Kähler diamond is symmetric about both the horizontal and the vertical axis.. For example, the Hodge diamond of $\mathbb{P}^2$ looks like this:

$$\begin{pmatrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{pmatrix}\ . \tag{2.21}$$

2.1.3 Fiber Bundles

Fiber and Vector Bundles. A fiber bundle $F \to E \xrightarrow{\pi} M$ consists of the following data:

i) A differentiable manifold M, called *base space*.

ii) A differentiable manifold F, called the *fiber*.

iii) A differentiable manifold E, called *total space*, together with a projection $\pi : E \to M$.

iv) A *local trivialization*: There has to be an atlas of M such that for every chart $U \subset M$,

$$E|_U \cong U \times F\ . \tag{2.22}$$

One example is the trivial bundle $E = M \times F$.

A *section* s of a fiber bundle $\pi : E \to M$ is a map $s : M \to E$ with $\pi \circ s = \mathrm{id}_M$. The space of sections is denoted by $\Gamma(E)$.

A *vector bundle* is a fiber bundle where the fiber F is a vector space. A vector field on a manifold is a section of the tangent bundle $TM = \bigcup_p T_pM$, the tangent bundle is a vector bundle. Each vector bundle admits a global section, the zero section $s : M \to 0 \in F$.

Holomorphic Vector Bundles. A vector bundle is called holomorphic if π as well as all local trivializations and transition functions are holomorphic. A prominent example is the holomorphic tangent bundle TX^+ of X. Its fibers are spanned by the vectors $\{\partial_\mu\}$ in a point p.

If the fiber of a holomorphic vector bundle has dimension one, it is called a holomorphic line bundle. For example, $\mathcal{O}_X$ is the sheaf of sections of the trivial holomorphic line bundle $X \times \mathbb{C}$, it is also called $\mathcal{O}$ or $\mathcal{O}_X$.

Example for holomorphic vector bundles also include the bundles of holomorphic r-forms $\Omega^{r,0}(X)$, and most importantly the *canonical bundle*

$$K_X = \Omega^{m,0}(X) \ . \tag{2.23}$$

The *adjunction formula* reads

$$K_Y \cong K_{X|_Y} \otimes \det \mathcal{N}_{Y/X} \ , \tag{2.24}$$

where $\mathcal{N}_{Y/X}$ is the normal bundle of Y in X.

The holomorphic line bundles over X, together with the tensor product $\otimes$, form a group. This group is called the Picard group $\mathrm{Pic}(X)$.

Divisors of Complex Manifolds. The group of divisors of X is the free Abelian group over all irreducible hypersurfaces of X, it is called $\mathrm{Div}(X)$. There is a natural homomorphism $\mathcal{O} : \mathrm{Div}(X) \to \mathrm{Pic}(X)$, essentially assigning to a hypersurface Y its normal bundle: $\mathcal{O}(Y)|_Y = \mathcal{N}_{Y/X}$. This lets us rewrite the adjunction formula,

$$K_Y \cong (K_X \otimes \mathcal{O}(Y))|_Y \ . \tag{2.25}$$

A divisor D with $\mathcal{O}(D) = \mathcal{O}$ is called *principal*, we often consider divisor classes under the equivalence relation where D and D' are equivalent if and only if $D - D'$ is principal.

We can also characterize the image of $\mathcal{O}$ in $\mathrm{Pic}(X)$: If a line bundle L has a section s that is not identically zero, its zero set (including multiplicities) $Z_L(s)$ defines a divisor such that $\mathcal{O}(Z_L(s)) = L$. If L has no such section, it is not contained in the image of $\mathcal{O}$.

2.1.4 Chern Classes

Definition. Let B be a matrix, then $\det(\mathrm{id} + B) = 1 + \mathrm{tr}\, B + \cdots = \sum_k P_k(B)$, where P_k is a polynomial of degree k. The k-th *Chern class* of a complex vector bundle $\pi : E \to M$ is

$$\mathrm{c}_k(E) = \left[P_k \left(\frac{\mathrm{i}}{2\pi} F \right) \right] \in H^{2k}(M) \ . \tag{2.26}$$

$F \in \Omega^2(M) \otimes \mathrm{End}(E)$ is the curvature of an arbitrary connection (see section A.5) on E, $\mathrm{c}_k(E)$ does not depend on the choice of F. For example, $\mathrm{c}_1(E) = \frac{\mathrm{i}}{2\pi} [\mathrm{tr}\, F]$.

We define the Chern classes of a complex manifold X to be those of the holomorphic tangent bundle:

$$\mathrm{c}_k(X) = \mathrm{c}_k(TX^+) \ . \tag{2.27}$$

A related concept are the Chern characters $\mathrm{ch}_k(E)$. To define those, we just have to replace $\det(\mathrm{id} + B)$ by $\mathrm{tr}(\mathrm{e}^B)$ and get $\mathrm{ch}_k(E) = \frac{1}{k!} \mathrm{tr} \left[\frac{\mathrm{i}}{2\pi} F \right]^k$.

Properties of c_1.

- Let E^* be the dual bundle of E, then

$$\mathrm{c}_1(E^*) = -\,\mathrm{c}_1(E) \ . \tag{2.28}$$

  For two bundles E_1 and E_2,

$$\mathrm{c}_1(E_1 \oplus E_2) = \mathrm{c}_1(E_1) + \mathrm{c}_1(E_2) \ . \tag{2.29}$$

  If both are line bundles, then also

$$\mathrm{c}_1(E_1 \otimes E_2) = \mathrm{c}_1(E_1) + \mathrm{c}_1(E_2) \ . \tag{2.30}$$

- The first Chern class of the canonical bundle is
$$\mathrm{c}_1(K_X) = -\,\mathrm{c}_1(X)\ , \tag{2.31}$$
it is called the canonical class $\mathcal{K}_X = \mathrm{c}_1(K_X)$.
- We will later define the tautological line bundle $\mathcal{O}(1)$ over $\mathbb{P}^n$. Chern classes are normalized:
$$\int_{\mathbb{P}^n} \mathrm{c}_1(\mathcal{O}(1))^n = 1\ . \tag{2.32}$$

If now $Y \subset X$ is an irreducible hypersurface and $i : Y \to X$ the inclusion, then we can rewrite the adjunction formula again,
$$\mathrm{c}_1(Y) = i^*\left(\mathrm{c}_1(X) - \mathrm{c}_1(\mathcal{O}(Y))\right)\ . \tag{2.33}$$
Also, let $[Y]$ be the Poincaré dual class, then
$$\mathrm{c}_1(\mathcal{O}(Y)) = [Y]\ . \tag{2.34}$$

Calabi-Yau Manifolds. There are several different definitions of Calabi-Yau manifolds, which are not always equivalent. The main point is that CY manifolds are compact Kähler manifolds and have a vanishing Ricci-tensor, or equivalently a vanishing first Chern class. For a more in-depth discussion, see subsection B.4.4 in the mathematical appendix.

The most general Hodge diamond of a CY threefold is
$$\begin{pmatrix} &&&1&&& \\ &&0&&0&& \\ &0&&b^{1,1}&&0& \\ 1&&b^{2,1}&&b^{2,1}&&1 \\ &0&&b^{1,1}&&0& \\ &&0&&0&& \\ &&&1&&& \end{pmatrix}\ . \tag{2.35}$$

2.2 Projective Geometry

2.2.1 Line Bundles and Divisors

The Tautological Line Bundle $\mathcal{O}(1)$. $\mathcal{O}(1)$ is a line bundle over $\mathbb{P}^n$, its global sections are polynomials of degree 1 in the homogeneous coordinates. We define the homomorphism $\mathcal{O} : \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n)$ by
$$\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}\ . \tag{2.36}$$
The sections of $\mathcal{O}(k)$ are polynomials of degree k.

In fact, the line bundles $\mathcal{O}(k)$ are all holomorphic line bundles over $\mathbb{P}^n$. For example, the canonical bundle of projective space is
$$K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)\ . \tag{2.37}$$

Divisors. $\mathbb{P}^n$ has only one divisor class, the hyperplane class Y. The two previously given definitions of $\mathcal{O}$ are compatible: Let D be a divisor and $\deg D$ its degree. The diagram
$$\begin{tikzcd} \mathrm{Div}(\mathbb{P}^n)/\!\sim \arrow[rr, "\deg", "\sim"'] \arrow[dr, "\mathcal{O}"] && \mathbb{Z} \arrow[dl, "\mathcal{O}"'] \\ & \mathrm{Pic}(\mathbb{P}^n) & \end{tikzcd} \quad \text{commutes.} \tag{2.38}$$

Chern Classes. $H^2(\mathbb{P}^n) = H^{1,1}(\mathbb{P}^n)$ has only one generator, we call it $J = [Y] = \mathrm{c}_1(\mathcal{O}(1))$. Important Chern classes are:

$$\mathrm{c}_1(\mathcal{O}(k)) = k\,J \qquad \text{and} \tag{2.39}$$

$$\mathrm{c}_1(\mathbb{P}^n) = (n+1)\,J\ . \tag{2.40}$$

2.2.2 Constructing Calabi-Yaus

Complete Intersections. Consider the product space $X = \mathbb{P}_1^{n_1} \times \cdots \times \mathbb{P}_m^{n_m}$. Such a space has m divisor classes, the hyperplane classes of the individual $\mathbb{P}_r^{n_r}$, and its first Chern class is

$$\mathrm{c}_1(X) = \sum_r (n_r + 1)\ . \tag{2.41}$$

A *complete intersection* is the intersection of the vanishing locus of k polynomials, call the degrees of homogeneity of the a-th such polynomial q_a^r. Such a configuration is summarized in the matrix

$$\left[\boldsymbol{n}\,\middle\|\,\underline{q}\right] = \left[\begin{array}{c\|ccc} n_1 & q_1^1 & \cdots & q_k^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_m & q_1^m & \cdots & q_k^m \end{array}\right]\ . \tag{2.42}$$

The canonical bundle of the complete intersection is $\bigotimes_r \mathcal{O}_r\left(\sum_a q_a^r - (n_r+1)\right)$, and accordingly its first Chern class is

$$\mathrm{c}_1\left[\boldsymbol{n}\,\middle\|\,\underline{q}\right] = \sum_r \left(n_r + 1 - \sum_a q_a^r\right) J_r\ . \tag{2.43}$$

When we want to compute the Hodge numbers of this kind of manifolds, the *Lefshetz hyperplane theorem* is often extremely useful: Let X be a compact complex manifold and $Y \subset X$ a smooth hypersurface with ample normal bundle $\mathcal{O}(Y)$. Then, $i^* : H^q(X) \to H^q(Y)$ is an isomorphism for $q < \dim_{\mathbb{C}} Y$ and injective for $q = \dim_{\mathbb{C}} Y$.

Weighted Projective Spaces. In the definition (2.5) of ordinary projective space $\mathbb{P}^n$, all the coordinates scale with the same weight. We'll now modify this and let the quasi-homogeneous coordinates scale with weights

$$\begin{array}{cccc} z_0 & z_1 & \cdots & z_n \\ \hline w_0 & w_1 & \cdots & w_n \end{array}\ , \tag{2.44}$$

i.e. $[z_0 : \cdots : z_n] = [\Lambda^{w_0} z_0 : \cdots : \Lambda^{w_n} z_n]$. The locus where all of $\langle z_0 \cdots z_n \rangle$ is still excluded from the manifold. The so-defined manifold is called *weighted projective space* $\mathbb{P}^n_{(w_0:\cdots:w_n)}$.

Weighted projective spaces are projective manifolds. There is only one divisor class and the quasi-homogeneous coordinates z_i are sections of $\mathcal{O}(w_i)$. The canonical bundle of weighted projective space is $K = \mathcal{O}(-\sum_i w_i)$ and the first Chern class is

$$\mathrm{c}_1\left(\mathbb{P}^n_{(w_0:\cdots:w_n)}\right) = \sum_i w_i\,J\ . \tag{2.45}$$

A polynomial in the quasi-homogeneous coordinates is said to have degree q_a if it is a section of $\mathcal{O}(q_a)$, i.e. if it scales with Λ^{q_a}. The first Chern class of the complete intersection V of polynomials with degrees q_a is

$$\mathrm{c}_1(V) = \left(\sum_i w_i - \sum_a q_a\right) J\ . \tag{2.46}$$

2.2.3 Blowing Up

Singularities. In many cases, we will be working with singular varieties. Singularities can arise in different ways:

- The embedding space can have singularities to begin with. For example, weighted projective spaces are singular in general.
- A complete intersection defined by some polynomials F_a becomes singular in a locus where the defining equations are not regular, i.e. $F_a = 0$ and $\mathrm{d}F_a = 0$ at the same time.
- Often, we like to take quotient manifolds under some group action. The resulting orbifold has singularities in those points that are fixed under the group action.

Blow-Up. Singularities can usually be resolved by blowing them up. Blowing up an n-dimensional manifold in a point $x \in X$ essentially means to replace that point with $\mathbb{P}^{n-1}$. More precisely, there is a holomorphic map σ from the blow-up $\hat{X}$ to X such that

i) the exceptional divisor $E = \sigma^{-1}(x)$ is isomorphic to $\mathbb{P}^{n-1}$, and

ii) $\hat{X}\backslash E$ and $X\backslash\{x\}$ are isomorphic.

For example, the blow-up of $\mathbb{C}^n$ in 0 is $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ viewed as a projective manifold (see subsection C.2.2). That also how we can construct the blow-up in general: Blowing up is a local operation and the complex manifold X always looks like $\mathbb{C}^n$ in a neighborhood of x.

The canonical bundle changes under blow-up:

$$K_{\hat{X}} \cong \sigma^* K_X \otimes \mathcal{O}_{\hat{X}}\left((n-1)E\right) \ . \tag{2.47}$$

2.2.4 Toric Varieties

Basic Idea. Let us consider, for example, the product X of two weighted projective spaces. X can be described using quasi-homogeneous coordinates scaling as

$$\begin{array}{cccccc} z_1 & \cdots & z_m & z_1' & \cdots & z_n \\ \hline w_1 & \cdots & w_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & w_1' & \cdots & w_n' \end{array} \ . \tag{2.48}$$

The sets of coordinates which are not allowed to vanish at the same time can be represented as a monomial ideal $\langle z_1 \cdots z_m, z_1' \cdots z_n' \rangle$. X has two divisor classes A and B corresponding two the two rows, we easily get the first Chern class by summing up the entries of the table:

$$\mathrm{c}_1(X) = \left(\sum_i w_i\right)[A] + \left(\sum_j w_j'\right)[B] \ . \tag{2.49}$$

Note that in homology, $[A]^m = 0 = [B]^n$.

We generalize this concept and let coordinates scale in more than one relation, expressed by the general table

$$\begin{array}{ccccc} z_1 & \cdots & z_k & \cdots & z_n \\ \hline w_1^1 & \cdots & w_k^1 & \cdots & w_n^1 \\ \vdots & & \vdots & & \vdots \\ w_1^m & \cdots & w_k^m & \cdots & w_n^m \end{array} \ . \tag{2.50}$$

Again, a monomial ideal called the *SR ideal* defines the sets of quasi-homogeneous coordinates that can not vanish simultaneously. The resulting space is a *toric variety* T.

T has m divisor classes $A_1, \dots, A_m$ corresponding to the rows of the table. The first Chern class is just $\mathrm{c}_1(T) = \sum_{i,k} w_k^i\,[A_i]$. The vanishing locus of a coordinate z_k defines a divisor $\{z_k\}$ of the class $\sum_i w^i\,A_i$, its Poincaré dual is

$$[z_k] = \sum_i w^i\,[A_i]\ . \tag{2.51}$$

The SR ideal induces additional relations in homology: If the ideal contains the monomial $z_{k_1}\cdots z_{k_\ell}$, then $\bigwedge_{a=1}^{\ell}[z_{k_a}] = 0$.

Blow-Up of Projective Space. This gives us an easier way of describing the blow-up of $\mathbb{P}^2$, say. Let x, y and z be the homogeneous coordinates of $\mathbb{P}^2$, then the blow-up in $y = z = 0$ has a new coordinate λ scaling

$$\begin{array}{cccc} x & y & z & \lambda \\ \hline 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array} \tag{2.52}$$

and the SR ideal $\langle xyz, yz, x\lambda\rangle = \langle yz, x\lambda\rangle$. The blow-up map σ is given by $\sigma(x, y, z, \lambda) = [x : \lambda y : \lambda z]$.

In general, we can write down the blow-up of a toric variety T in the locus where a set of coordinates $\{\zeta_k\}$ vanishes. See section D.1 to see the algorithm written out in actual code, in short: We add a new coordinate λ and a new row in the scaling relations containing 1 for all ζ_k, -1 for λ and zeros otherwise. We take the old SR ideal and add the following generators:

- The singular locus $\zeta_1 \cdots \zeta_N$.
- For each old generator containing one or more ζ_k, we remove all ζ_k and add λ.

Actual Definition. Our viewpoint so far has been very basic. In mathematical literature, one usually considers the toric variety of a fan. We will quickly describe what that means, following [10, Ch. A.2].

We start with a finite set of vectors $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ in $\mathbb{R}^N$, called rays. The convex span of a subset of linear independent rays defines a *strongly convex rational polyhedral cone* c if $-c \cup c = \{0\}$. A set of cones is a *fan* if all faces of each cone are contained in the fan as well, and if every intersection of two cones is a common face of both. The fan is defined uniquely by listing the rays and the maximal cones (which are not faces of other cones).

We get the toric variety T corresponding to a fan in the following way: The quasi-homogeneous coordinates correspond to the rays of the fan. Each linear relation $\sum_k w_k v_k = 0$ between the rays corresponds to a scaling relation with weights $(w_1 : \cdots : w_n)$. There are $m = n - N$ such relations and the dimension of T is therefore N. Finally, a monomial is contained in the SR ideal of T if and only if the corresponding rays do not form a cone in the fan.

2.3 Theory of SU(N)

We will follow [11] to recap some properties of SU(N) Lie groups and their associated Lie algebras.

2.3.1 Definitions

We define the *special unitary group* SU(N) to be the set of unitary matrices with determinant one. It is a simply connected compact Lie group of dimension $N^2 - 1$. As such, every $U \in \mathrm{SU}(N)$ can be written as $U = \exp(\mathrm{i}h)$ for an element $h \in \mathfrak{su}(N)$ of its Lie algebra. By plugging $U(t) = \exp(\mathrm{i}ht)$ into the defining relations $UU^\dagger = 1$ and $\det U = 1$, and differentiating at $t = 0$, we get

$$\mathfrak{su}(N) = \left\{h \in \mathbb{C}^{N\times N} : h = h^\dagger, \operatorname{tr} h = 0\right\}\ . \tag{2.53}$$

We choose the following basis of $\mathfrak{su}(N)$: Let 1_{ab} be the matrix which is zero except for the element in the a-th row and b-th column, which is one: $(1_{ab})_{ij} = \delta_{ai}\delta_{bj}$. Then we define the generators

$$E_{ab}^{(1)} = \frac{1}{2}\left(1_{ab} + 1_{ba}\right), \quad E_{ab}^{(2)} = -\frac{\mathrm{i}}{2}\left(1_{ab} - 1_{ba}\right) \qquad \text{and} \qquad H_a = 1_{aa} - \frac{1}{n}\mathbb{1}\ . \tag{2.54}$$

Since we can restrict to $1 \le a < b \le N$ for the E_{ab} and since $H_N = -\sum_{a=1}^{N-1} H_a$, this makes $N(N-1)+(N-1) = N^2-1$ independent generators.

Note that in the familiar $\mathfrak{su}(2)$ case, $\sigma_1 = 2E_{12}^{(1)}$, $\sigma_2 = 2E_{12}^{(2)}$ and $\sigma_3 = 2H_1$.

Alternative Choices of Basis. The subalgebra spanned by the H_a is of special interest. It is a *Cartan subalgebra*, that is a maximal toral[1] Lie subalgebra of $\mathfrak{su}(N)$, and hence Abelian. We will come back to the significance of this subalgebra later, here we only mention some alternative choices of basis.

One such choice that sometimes is preferred is

$$H'_a = \frac{1}{\sqrt{2(a+a^2)}} \left(\sum_{k=1}^{a} \mathbb{1}_{kk} - a\,\mathbb{1}_{(a+1)(a+1)} \right) . \tag{2.55}$$

With this choice, the set of all generators $\{T^a\} = \{E_{ab}^{(1)}\} \cup \{E_{ab}^{(2)}\} \cup \{H'_a\}$ satisfies the normalization relation

$$\mathrm{tr}(T^a \cdot T^b) = \frac{1}{2}\delta^{ab} . \tag{2.56}$$

Another possibility is

$$T^{\alpha_i} = 1_{ii} - 1_{(i+1)(i+1)} \tag{2.57}$$

for $i = 1, \ldots, (N-1)$, this one is useful because the scalar products

$$\mathrm{tr}\left(T^{\alpha_i} \cdot T^{\alpha_j}\right) = C_{ij} \tag{2.58}$$

form the *Cartan matrix* C_{ij} of $\mathfrak{su}(N)$. We will later see that the Cartan matrix is given by the scalar products $C_{ij} = \langle \alpha_i, \alpha_j \rangle$ of the *simple roots* α_i of the Lie algebra. Note that we can add T^{α_0} and $\mathrm{tr}\left(T^{\alpha_i} \cdot T^{\alpha_j}\right)$ forms the extended Cartan matrix $\tilde{C}_{ij}$.

The Cartan matrix of $\mathfrak{su}(N)$ has 2 on its diagonal and -1 next to the diagonal, for example for $\mathfrak{su}(5)$

$$C_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \tilde{C}_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix} . \tag{2.59}$$

2.3.2 Irreducible Representations

By definition, $\mathrm{SU}(N)$ acts in the *fundamental representation* on $V = \mathbb{C}^N$. This immediately gives us representations of $\mathrm{SU}(N)$ on $V^{\otimes k}$ by $U(v \otimes w) = Uv \otimes Uw$.

Let us take a slightly different viewpoint and choose a basis in V: A vector $v \in V$ is then given by its components v^i and the action of $U \in \mathrm{SU}(N)$ is given by a matrix U^i_j: $v^i \mapsto U^i_j v^j$. In general, a tensor in $V^{\otimes k}$ has k indices, e.g. $T^{i_1 \cdots i_k}$ and the action of U is

$$T^{i_1 \cdots i_k} \mapsto U^{i_1 \cdots i_k}_{j_1 \cdots j_k} T^{j_1 \cdots j_k} \tag{2.60}$$

with $U^{i_1 \cdots i_k}_{j_1 \cdots j_k} = U^{i_1}_{j_1} \cdots U^{i_k}_{j_k}$.

These representations are not irreducible. For example, the representation $V \otimes V$ decomposes into a symmetric and an antisymmetric part:

$$V \otimes V = (V \odot V) \oplus (V \wedge V) . \tag{2.61}$$

Those representations are irreducible. In general, we get irreducible representations by symmetrizing and antisymmetrizing indices. Which pairs of indices are symmetric and which are antisymmetric can be visualized with *Young tableaux*.

Theorem (Fundamental Theorem)**.** *There is a one-to-one correspondence between irreducible representations of* $\mathrm{SU}(N)$ *and Young tableaux of no more than* $(N-1)$ *rows.*

[1] A subalgebra is toral if all of its elements are semi-simple. An element x of a Lie algebra $\mathfrak{g}$ is semi-simple if the endomorphism $[x, \cdot]$ of the vector space $\mathfrak{g}$ is semi-simple. An endomorphism of a vector space is semi-simple if the orthogonal complement of every invariant subspace is again invariant.

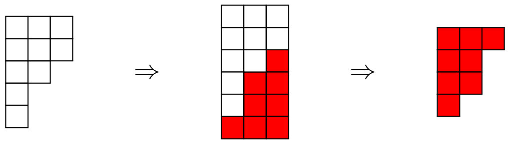


Figure 2.1: Visualization of a dual tableau in SU(6).

Young Tableaux. Young tableaux consist of a number of boxes arranged in rows, where the amount of boxes in the rows is non-increasing from top to bottom. Indices corresponding to boxes in the same row have to be symmetric, indices in the same column are antisymmetric. Let's consider some examples, taken from [12]:

- The empty Young tableau $\cdot$ stands for the trivial representation $V^{\otimes 0} = \mathbb{C}$.
- $\square$ denotes the fundamental representation $V = \mathbb{C}^N$, i.e. vectors v^i.
- $\square\square = V \odot V$ are symmetric matrices, $T_{ij} = T_{ji}$.
- $\begin{smallmatrix}\square\\ \square\end{smallmatrix} = V \wedge V$ are antisymmetric matrices, $T_{ij} = -T_{ji}$.
- $\begin{smallmatrix} i & j \\ k & \end{smallmatrix}$ stands for objects T_{ijk} with three indices, $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{kji}$.
- The adjoint representation is the representation on $\mathfrak{su}(N)$ given by the action $U.h = UhU^{-1}$. It corresponds to the tableau with $(N-1)$ rows of lengths $(2, 1, \ldots, 1)$.
- The dual representation (see definition A.69) of a given tableau with row lengths $(f_1, f_2, \ldots, f_n = 0)$ has row lengths $(f_1 - f_n = f_1, f_1 - f_{n-1}, \ldots, f_1 - f_1 = 0)$. For a visualization see figure 2.1.

Note. The classification of SO(N) works in almost the same way, except for one problem: One can further reduce representations by taking the trace. For example, $\square\square = \square\square_{\text{traceless}} \oplus \mathbb{C}$.

Note. In Physics, a representation is usually denoted by its dimension, e.g. $\square = \mathbf{N}$.

2.3.3 Algebra Representations

Representations of $\mathfrak{su}(2)$. In $\mathfrak{su}(2)$ we can classify the irreducible representations easily using a different approach, which should be familiar (see e.g. [13]): Let $H = H_1$, $E = E_{12}^{(1)}$ and $F = E_{12}^{(2)}$ be the generators, their commutation relations are

$$[H, E] = \mathrm{i}F, \quad [H, F] = -\mathrm{i}E \quad \text{and} \quad [E, F] = \mathrm{i}H\ . \tag{2.62}$$

We define raising and lowering operators $J^{\pm} = E \pm \mathrm{i}F$ such that $[H, J^{\pm}] = \pm J^{\pm}$, they change the eigenvalue of a state with respect to H by ± 1.

We choose an eigenbasis of H and define the *weight* of a state to be its eigenvalue. After a bit of thinking, we see that an irreducible representation is uniquely characterized by its heighest weight w, which has to be integer or half-integer. The dimension of this representation is $(2w+1)$ spanned by vectors with weights $-w, -w+1, \ldots, w-1, w$.

Representations of $\mathfrak{su}(N)$. In the general case we can do something similar.

Let us first talk about the commutation relations of $\mathfrak{su}(N)$. First of all, all of the H_a commute with each other. We want to use raising and lowering operators like above, define

$$J_{ab}^{\pm} = E_{ab}^{(1)} \pm \mathrm{i}E_{ab}^{(2)} \tag{2.63}$$

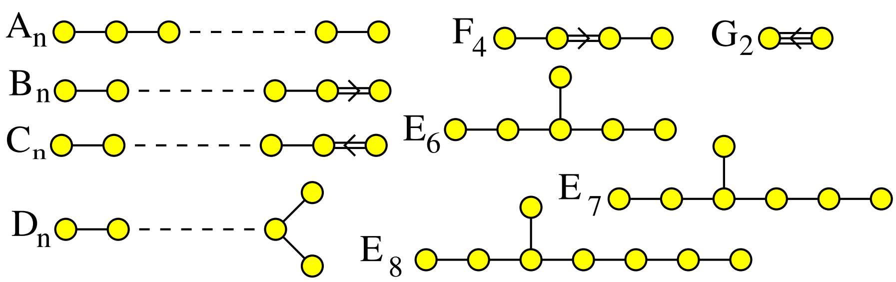


Figure 2.2: Finite connected Dynkin diagrams [15].

for $a < b$. Calculation shows that

$$[\boldsymbol{H}, J^+_{ab}] = \left(\boldsymbol{e}^{(a)} - \boldsymbol{e}^{(b)}\right) J^+_{ab} \qquad \text{and} \qquad [\boldsymbol{H}, J^-_{ab}] = -\left(\boldsymbol{e}^{(a)} - \boldsymbol{e}^{(b)}\right) J^-_{ab} \tag{2.64}$$

where $\boldsymbol{e}^{(i)}$ are the unit vectors in $\mathbb{R}^N$. The vectors $\boldsymbol{e}^{(a)} - \boldsymbol{e}^{(b)}$ are called the *roots* of $\mathfrak{su}(N)$.

In a given irreducible representation, we can simultaneously diagonalize $\boldsymbol{H}$. A state in this representation is characterized by its vector of eigenvalues, called its *weight*. Note that all those weights $(w_1, \dots, w_n)$ satisfy $\sum_a w_a = 0$, they lie in an $\mathbb{R}^{N-1}$ subspace.

The raising and lowering operators change the weight of a state by the corresponding root. It turns out that an irreducible representation is uniquely characterized by its highest weight: The highest weight is the weight of that state u where $J^+_{ab} u = 0$ for all $a < b$. In [11] there are some examples for highest weights of representations corresponding to Young tableaux.

Note on Semi-Simple Lie Algebras [14]. This procedure is much more general and powerful: Let $\mathfrak{g}$ be a Lie algebra and H the Cartan subalgebra (i.e. a maximal toral one, see subsection 2.3.1), which is abelian. We let H act on $\mathfrak{g}$ in the adjoint representation, $\mathfrak{g}$ decomposes in simultaneous eigenspaces:

$$\mathfrak{g} = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \,. \tag{2.65}$$

Here $\alpha \in H^*$ are the eigenvalues and $L_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)\, x\}$ the eigenspaces. The elements of Φ are called the roots of $\mathfrak{g}$. The classification of semi-simple Lie algebras is achieved by analysing all possible root systems in the following way:

There is a special choice of basis for the root system called the *simple roots* of $\mathfrak{g}$. The number of simple roots of $\mathfrak{g}$ is called the *rank* $\operatorname{rk}\mathfrak{g}$ of the algebra, we'll call the simple roots $\alpha_1, \dots, \alpha_{\operatorname{rk}\mathfrak{g}}$. Define $\langle \alpha_i, \alpha_j \rangle = 2\frac{x \cdot y}{x \cdot x}$, the matrix $C_{ij} = \langle \alpha_i, \alpha_j \rangle$ is the *Cartan matrix* of the algebra. Its entries are highly constrained: For example, the diagonal elements are always 2 and the off-diagonal elements can only take on the values -1, -2 or -3.

In fact, all possible Cartan matrices are classified by the *Dynkin diagrams*, see figure 2.2: The nodes correspond to the simple roots of the algebra, and if $C_{ij} = -k$ we draw k lines between two nodes. (If the Cartan matrix is not symmetric, we draw an an arrow on the line.) Since the Cartan matrix uniquely determines the Lie algebra, this is a classification of Lie algebras as well. As we have seen before, the algebras $\mathfrak{su}(N)$ have rank $\operatorname{rk}\mathfrak{su}(N) = N - 1$ and correspond to the A_{N-1} Dynkin diagrams. Note that figure 2.2 only contains connected Dynkin diagrams corresponding to simple Lie algebras, diagrams of semi-simple Lie algebras can be disconnected.

Representations of semi-simple Lie algebras can be classified similarly: Every given representation V decomposes into simultaneous eigenspaces under the action of H:

$$V = \bigoplus_\chi V_\chi \tag{2.66}$$

for some $\chi \in H^*$ called the weights of the representation and $V_\chi = \{v \in V : h.v = \chi(h)\, v\}$. We sort the roots in positive and negative ones (Φ^+ and Φ^-) and define highest weight vectors to be $v \in V$ where $L_\alpha.v = 0$ for $\alpha \in \Phi^+$. Continuing in this direction leads to a general classification of representations of semi-simple Lie algebras.

2.3.4 Advanced Topics

One important question arising quite often is: The *tensor product* of two irreducible representations is in general not irreducible any more. How does it decompose into irreducible representations?

The answer to this questions can be worked out using Young tableaux in full generality. The process is quite tedious, however, and can be looked up in [11, Ch. V].

Another important topic is *subduction*: If a representation of SU(N) is given, then it is also a representation of a subgroup $G < \mathrm{SU}(N)$. The question we will be asking quite often in the following is: Which representation?

Unfortunately, answering this question can be quite difficult in general. However, in [11, Ch. VI] this is worked out for the subgroup $\mathrm{SU}(N) \times \mathrm{SU}(M)$ of $\mathrm{SU}(NM)$ or of $\mathrm{SU}(N + M)$ which covers the most important cases for us.

In simple cases it is quite clear though: Take the subgroup $\mathrm{SU}(N) \times \mathrm{SU}(M)$ of $\mathrm{SU}(N + M)$ and consider the fundamental representation of $\mathrm{SU}(N + M)$. Choose the basis such that SU(N) acts on the upper N components and SU(M) on the lower M components of the vector. Then we obviously get

$$\square_{N+M} \to (\mathbf{N}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{M}) \; . \tag{2.67}$$

Chapter 3

Basics of Physics

3.1 Particle Physics

3.1.1 The Standard Model

The *Standard Model of particle physics* is a Yang-Mills theory as described in subsection A.5.4. Its gauge group is $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y$, the subscript Y stands for the hypercharge. Note that there are three a priori different coupling constants associated to the three factors: The *strong coupling* g_s of SU(3), the *weak coupling* g of SU(2) and the *hypercharge coupling* g' corresponding to $\mathrm{U}(1)_Y$.

The Standard Model has the following particle content [16, Ch. 22.4] [17, Ch. 6]:

Fermionic matter: There are three *generations* of

Symbol	Name	SU(3)-rep.	SU(2)-rep.	$\mathrm{U}(1)_Y$-charge
Q_L	Left-handed quark	$\mathbf{3}$	$\mathbf{2} = (u_L, d_L)$	$1/6$
u_R	Right-handed up-type quark	$\mathbf{3}$	$\mathbf{1}$	$2/3$
d_R	Right-handed down-type quark	$\mathbf{3}$	$\mathbf{1}$	$-1/3$
L_L	Left-handed lepton	$\mathbf{1}$	$\mathbf{2} = (e_L, \nu_L)$	$-1/2$
e_R	Right-handed lepton	$\mathbf{1}$	$\mathbf{1}$	-1

Note: If the fermion ψ has a charge Y under $\mathrm{U}(1)_Y$, that means that we take $Y \in \mathbb{R} = \mathfrak{u}(1)$ as the generator T in gauge transformations.

Gauge bosons: For each factor in G we have *gauge bosons*, which are the components of the vector potential. The gauge bosons of SU(3) are called *gluons* G^a_μ $(a = 1, \dots, 8)$, meaning that we write the gauge potential G of SU(3) as $G = G^a_\mu \frac{\lambda^a}{2} \mathrm{d}x^\mu$.[1]

The gauge bosons of SU(2) are named $W = W^a_\mu \frac{\sigma^a}{2} \mathrm{d}x^\mu$ and the gauge boson of $\mathrm{U}(1)_Y$ is $B = B_\mu \mathrm{d}x^\mu$. Summarizing:

Symbol	Name	SU(3)-rep.	SU(2)-rep.	Transformation under $\mathrm{U}(1)_Y$[2]
G	Gluon	$\mathbf{8}$	$\mathbf{1}$	trivial
W	W-Bosons	$\mathbf{1}$	$\mathbf{3}$	trivial
B	B-Boson	$\mathbf{1}$	$\mathbf{1}$	given by (A.84)

Higgs field: Additionally, there is one complex scalar field:

Symbol	Name	SU(3)-rep.	SU(2)-rep.	$\mathrm{U}(1)_Y$-charge
H	Higgs	$\mathbf{1}$	$\mathbf{2} = (H^+, H^0)$	$1/2$

H acquires a non-zero vacuum expectation value because of how its potential is shaped. This has the consequence that the $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ symmetry is *spontaneously broken* down to the electromagnetic $\mathrm{U}(1)_{em}$.

[1] λ^a are the *Gell-Mann matrices*, a basis of $\mathfrak{su}(3)$. The $\frac{\lambda^a}{2}$ are the basis defined in (2.55).

[2] The actual $\mathrm{U}(1)_Y$ charge of all those fields is zero.

According to general theory, this makes one of the four real degrees of freedom of H massive. By choosing a suitable gauge, the other three degrees of freedom vanish and make three of the vector bosons massive. The massive ones are the *$W^{\pm}$-bosons* $W^{\pm} = \frac{1}{\sqrt{2}}(W^1 \mp \mathrm{i}W^2)$ and the *Z-boson* $Z = \cos\theta_W W^3 - \sin\theta_W B$. θ_W is called the *Weinberg angle* and can be calculated to be

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}} . \tag{3.1}$$

The *photon* $A = \sin\Theta_w W^3 + \cos\Theta_w B$ remains massless. At low energies this looks effectively like a U(1) gauge theory, called the electromagnetic gauge theory $\mathrm{U}(1)_{\mathrm{em}}$. Its coupling constant is the *electric charge* $e = \frac{gg'}{\sqrt{g^2+g'^2}}$ and the charge of a particle under $\mathrm{U}(1)_{\mathrm{em}}$ can be calculated as $Q = T_3 + Y$.

Now we understand the reasoning behind defining $W^{\pm}$: We use this basis instead of $W^{1,2}$ because $W^{\pm}$ are eigenstates of T_3 with defined charge $Q = \pm 1$.

3.1.2 Supersymmetry

There are good theoretical reasons to expect that nature exhibits *supersymmetry* (SUSY): It solves the hierarchy / naturalness problem of the electroweak theory, it provides us with a natural candidate for what dark matter could be, and it improves gauge coupling unification (see subsection 3.2.1 below). In a supersymmetric theory, there is a bosonic *superpartner* for every fermion and vice versa. With the following particle content (in addition to the particles listed above) we get the *Minimal Supersymmetric Standard Model* (*MSSM*):

Sfermions: Sfermions are the spin-0-superpartners of fermions. There are three generations of

Symbol	Name	SU(3)-rep.	SU(2)-rep.	$\mathrm{U}(1)_Y$-charge
$\tilde{Q}_L$	Squark doublet	**3**	$\mathbf{2} = (u_L, d_L)$	1/6
$\tilde{u}_R$	Sup singlet	**3**	**1**	2/3
$\tilde{d}_R$	Sdown singlet	**3**	**1**	−1/3
$\tilde{L}_L$	Slepton doublet	**1**	$\mathbf{2} = (e_L, \nu_L)$	−1/2
$\tilde{e}_R$	Selectron singlet	**1**	**1**	−1

Gauginos: Gauginos are the spin-$\frac{1}{2}$-superpartners of gauge bosons. We have

Symbol	Name	SU(3)-rep.	SU(2)-rep.	Transformation under $\mathrm{U}(1)_Y$
$\tilde{G}$	Gluino	**8**	**1**	trivial
W	Winos	**1**	**3**	trivial
B	Bino	**1**	**1**	**1**

Higgsinos: With just one Higgs doublet, we cannot make the Yukawa terms of the Standard Model supersymmetry invariant. We need at least two different types of Higgs bosons to give mass to up-type and down-type quarks seperately. In the MSSM, we have two Higgs doublets and their superpartners, the Higgsinos:

Symbol	Name	SU(3)-rep.	SU(2)-rep.	$\mathrm{U}(1)_Y$-charge
H_d	Higgs	**1**	$\mathbf{2} = (H_d^0, H_d^-)$	−1/2
$\tilde{H}_d$	Higgsino	**1**	$\mathbf{2} = (\tilde{H}_d^0, \tilde{H}_d^-)$	−1/2
H_u	Higgs	**1**	$\mathbf{2} = (H_u^+, H_u^0)$	1/2
$\tilde{H}_u$	Higgsino	**1**	$\mathbf{2} = (\tilde{H}_u^+, \tilde{H}_u^0)$	1/2

Note that there can be different amounts of supersymmetry in a theory. In $d = 4$ spacetime dimensions, a spinor has four degrees of freedom and therefore there are at least four SUSY generators. This minimal case where every particle has only one superpartner is called $\mathcal{N} = 1$ supersymmetry – a theory with e.g. 32 SUSY generators would be $\mathcal{N} = 8$ SUSY. With *extended supersymmetry*, particles arrange in supermultiplets with more than 2 particles in one multiplet.

3.2 Grand Unified Theories

3.2.1 Motivation

Charge Quantization. As evident from the tables in section 3.1, the charge of particles under $\mathrm{U}(1)_Y$ appears to be quantized. Unfortunately, our discussion so far has not given us a theoretical reason for why it should be. The reason is ultimately that the groups U(1) and $(\mathbb{R}, +)$ (sometimes called "non-compact U(1)") have the same Lie algebra and we don't know which of those options is meant by "U(1)" in the gauge group.

If we have a compact U(1), then the charge is quantized, because there needs to be some n such that $\mathrm{e}^{2\pi n Y} = 1$ for all particle types. In the non-compact case, all values for Y are possible though. The suggested solution to this problem is to embed the Standard Model gauge group in a larger group, called the *GUT group*:

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y \subset G_{\mathrm{GUT}} \ . \tag{3.2}$$

If the U(1) subgroup is embedded in a non-abelian semi-simple Lie group, it is automatically compact [18]. This explains charge quantization. The larger symmetry must then be spontaneously broken down to the symmetry of the Standard Model at low energies.

Gauge Coupling Unification. If the GUT group is simple, having a grand unified theory explains the phenomenon of *gauge coupling unification* as well: A simple gauge group has only one coupling constant, hence we expect that physics at high energies can be described by only one coupling constant. At lower energies, the symmetry is broken and the three coupling constants of the Standard Model run individually according to the *renormalization group equation*, which reads

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha(M_Z^2)} + \frac{b}{4\pi} \ln \frac{Q^2}{M_Z^2} \tag{3.3}$$

at one-loop order. Here, $\alpha = \frac{g^2}{4\pi}$ where g is one of the coupling constants, Q^2 is the energy scale, M_Z^2 is the squared mass of the Z-boson and b is a constant (see below).

Therefore a GUT predicts that if we take the measured values $g_i(M_Z^2)$ for our three coupling constants $g_1 = \sqrt{5/3}g'$[3], $g_2 = g$ and $g_3 = g_s$ and run them to higher energies, they will eventually all agree. The scale where this happens is called the *GUT scale*.

Running Couplings in the Standard Model. Let's check whether gauge coupling unification happens in the Standard Model and in the MSSM. We need to calculate the constants b_i for all of the couplings and input the measured values $g_i(M_Z^2)$.

The constants are calculated from loop diagrams, the result is [17, 19]

$$b = \frac{11}{3} C_2(\mathrm{Ad}_G) - \sum_{\substack{\text{particle} \\ \text{type } p}} C(R_p) \cdot \begin{cases} 4/3 & \text{Dirac fermion} \\ 2/3 & \text{Chiral fermion} \\ 1/3 & \text{Complex scalar field} \end{cases} \tag{3.4}$$

where C is the Casimir and C_2 the second Casimir of a representation, Ad_G is the adjoint representation of the gauge group and R_p the representation of the particle type p.

In the special case of $G = \mathrm{SU}(N)$ we know that $C_2(\mathrm{Ad}_{\mathrm{SU}(N)}) = N$. Furthermore, the Casimir of the fundamental representation is $\frac{1}{2}$ and that of the adjoint representation is N. We can summarize this as

$$b_N = \frac{11N}{3} - \frac{1}{3} n_f - \frac{1}{6} n_s - \frac{2N}{3} n_{\mathrm{Ad}} \tag{3.5}$$

[3] The factor of $\sqrt{5/3}$ will become clear later: We have to include it because the hypercharge generator Y_1 embedded in SU(5) is normalized differently, precisely $Y_1 = \sqrt{3/5}\,Y$.

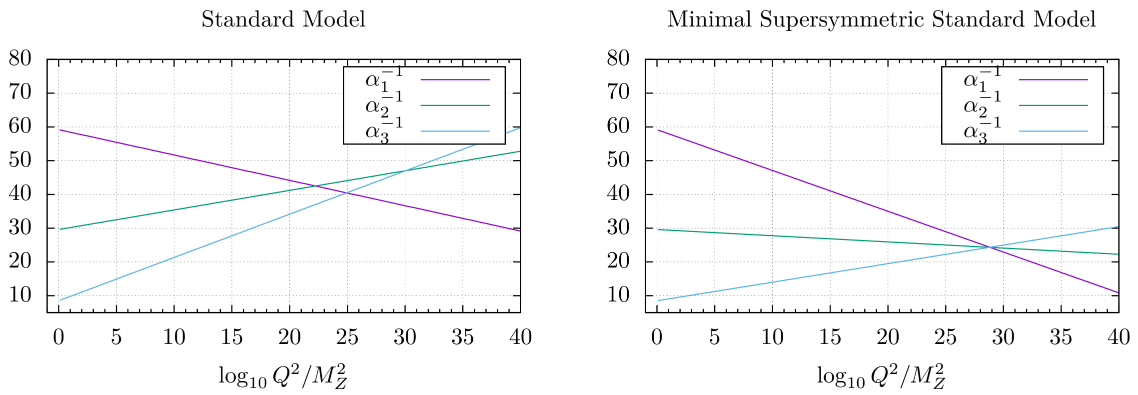


Figure 3.1: Running of the gauge couplings according to (3.3) and (3.7).

where n_f is the number of chiral fermions in the fundamental representation, n_s is the number of complex scalars and n_{Ad} is the number of fermions in the adjoint representation. Now we look the numbers up in section 3.1 and calculate for example

$$b_2^{\text{MSSM}} = \frac{11 \cdot 2}{3} - \frac{1}{3}(3 \cdot 4 + 2) - \frac{1}{6}(2 + 3 \cdot 4) - \frac{2 \cdot 2}{3} 1 = -1 \; .$$

Finally, we quote the result that for the $\mathrm{U}(1)_Y$-case:

$$b_1 = \frac{3}{5}\left(-\frac{2}{3}\sum_f y_f^2 - \frac{1}{3}\sum_s y_s^2\right) \; , \tag{3.6}$$

where y_f are the hypercharges of the fermions and y_s those of the scalars. For example,

$$b_1^{\text{SM}} = \frac{3}{5}\left[-2\left(6\frac{1}{6^2} + 3\frac{2^2}{3^2} + 3\frac{1}{3^2} + 2\frac{1}{2^2} + 1\right) - \frac{1}{3}\left(2\frac{1}{2^2}\right)\right] = -\frac{41}{10} \; .$$

All the values are:

$$\begin{aligned} b_1^{\text{SM}} &= -\frac{41}{10}, \quad & b_2^{\text{SM}} &= \frac{19}{6}, \quad & b_3^{\text{SM}} &= 7 \\ b_1^{\text{MSSM}} &= -\frac{33}{5}, \quad & b_2^{\text{MSSM}} &= -1, \quad & b_3^{\text{MSSM}} &= 3 \end{aligned} \tag{3.7}$$

Plugging in the experimental values $\alpha_1^{-1}(M_Z^2) = 59.2$, $\alpha_2^{-1}(M_Z^2) = 29.6$ and $\alpha_3^{-1}(M_Z^2) = 8.5$ we get figure 3.1. Already in the Standard Model, the curves almost meet – but in the MSSM, the curves meet within measure uncertainty. That is a strong argument in favour of supersymmetry as well as grand unification.

Note that calculations at higher loop order don't change this picture.

3.2.2 Overview: Possible GUTs

Georgi-Glashow Model (SU(5)): SU(5) is the smallest simple Lie group containing the Standard Model. All SM gauge fields are contained in the adjoint representation **24** of SU(5), under subduction

$$\mathbf{24} \to \underbrace{(\mathbf{8},\mathbf{1})_0}_{G} \oplus \underbrace{(\mathbf{1},\mathbf{3})_0}_{W} \oplus \underbrace{(\mathbf{1},\mathbf{1})_0}_{B} \oplus (\mathbf{3},\bar{\mathbf{2}})_{-5/6} \oplus (\bar{\mathbf{3}},\mathbf{2})_{+5/6} \tag{3.8}$$

(with the usual notation that the parentheses contain the $\mathrm{SU}(3) \times \mathrm{SU}(2)$ representation and the subscript gives the hypercharge).

One generation of fermions fits into a $\bar{\mathbf{5}}$ and a $\mathbf{10}$:

$$\begin{aligned} \bar{\mathbf{5}} &\to \overbrace{(\bar{\mathbf{3}},\mathbf{1})_{1/3}}^{d_R^c} \oplus \overbrace{(\mathbf{1},\mathbf{2})_{-1/2}}^{L_L} \\ \mathbf{10} &\to \underbrace{(\mathbf{3},\mathbf{2})_{1/6}}_{Q_L} \oplus \underbrace{(\bar{\mathbf{3}},\mathbf{1})_{-2/3}}_{u_R^c} \oplus \underbrace{(\mathbf{1},\mathbf{1})_1}_{e_R^c} \end{aligned} \tag{3.9}$$

In the first line we would expect $(\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{2}})$ but remember that $\mathbf{2} = \bar{\mathbf{2}}$ in SU(2).

For now, we'll assume that the GUT is broken to the Standard Model by a conventional symmetry breaking mechanism like a Lorentz-scalar field that takes a vacuum expectation value. In that case, the Georgi-Glashow model is already ruled out because the 12 extra gauge bosons in the **24** would lead to a proton lifetime too short to be compatible with experimental observations. Nevertheless, it is a good toy model for studying grand unification and we will have a closer look at it in subsection 3.2.3 for that reason. Also, in chapters 5 and 6 we will see that the mentioned problem can actually be avoided with a "stringy" breaking mechanism.

Pati-Salam Model ($\mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$): This was an earlier idea, appealing because it has left-right symmetry (at the GUT scale) and it doesn't predict proton decay. The gauge group is not simple, however, so that the model does not explain gauge coupling unification.

A family of fermions, including a right-handed neutrino[4] ν_R^c in representation $(\mathbf{1},\mathbf{1})_0$, fits as follows:

$$\begin{aligned} (\mathbf{4},\mathbf{2},\mathbf{1}) &\to (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{2})_{-1/2} \\ (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) &\to (\bar{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{1})_0 \ . \end{aligned} \tag{3.10}$$

In this theory, the SU(4) is broken to $\mathrm{SU}(3) \times \mathrm{U}(1)_{B-L}$ and then $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$ is broken down to $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$.

SO(10)**:** SO(10) is the largest simple gauge group which doesn't predict exotic fermions (i.e. fermions that are not in the Standard Model and not a right-handed neutrino). It contains SU(5) as a subgroup and all matter fits into a single irreducible representation, the spinorial **16**:

$$\mathbf{16} \to \mathbf{10} \oplus \bar{\mathbf{5}} \oplus \mathbf{1} \ . \tag{3.11}$$

The gauge bosons are contained in the adjoint representation **45**.

Note that this model predicts proton decay just like the Georgi-Glashow model, but it is not necessarily experimentally ruled out so far (it allows for a higher proton lifetime).

Note further that SO(10) contains the Pati-Salam model as well and

$$\mathbf{16} \to (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) \ . \tag{3.12}$$

Others: There is a large amount of other models with different gauge groups. For example, SU(6) contains SU(5) and one generation of fermionic matter is inside $\mathbf{15} \oplus \bar{\mathbf{6}} \oplus \bar{\mathbf{6}}$. In SU(8), we can put all fermionic matter into a single **64**, but then we get four generations of fermions. The exceptional Lie groups E_6, E_7 and E_8 are also often used.

However, there are a lot of problems with those theories (in fact, some of the problems already appear in the models mentioned above): Those theories predict generically objects like monopoles, domain walls or cosmic strings. They have a lot of extra gauge bosons and extra Higgs particles which have not been detected in nature. Often there is also a so-called doublet-triplet splitting problem (see below). A "simpler" GUT group would be much preferred from this point of view.

[4] Those are not included in the Standard Model but likely exist because of measured neutrino oscillations.

3.2.3 SU(5) GUT

Definitions. Let's have a closer look at how SU(5) is broken down to the Standard Model. We choose the Standard Model subgroup to be embedded in SU(5) in such a way that the SU(3) subgroup corresponds to the upper left 3×3 block and SU(2) to the lower right 2×2 block. In other words, the $\mathfrak{su}(3)$ subalgebra is generated by $E_{ab}^{(1,2)}$ for $1 \le a < b \le 3$ and two diagonal generators, e.g. $\tilde{H}_1 = \frac{1}{2}\left(1_{11} - 1_{33}\right)$ and $\tilde{H}_2 = \frac{1}{2}\left(1_{22} - 1_{33}\right)$. The $\mathfrak{su}(2)$ subalgebra is generated by $E_{45}^{(1,2)}$ and a third diagonal generator $\tilde{H}_3 = \frac{1}{2}\left(1_{44} - 1_{55}\right)$.

There is one more diagonal generator, commuting with both the $\mathfrak{su}(3)$ and the $\mathfrak{su}(2)$ subalgebras, this is the hypercharge generator. In proper normalization it is

$$Y_1 = \mathrm{diag}(-2,-2,-2,3,3)/\sqrt{60}\ , \tag{3.13}$$

because in Yang-Mills theory we want generators $\{T^a\}$ with $\mathrm{tr}\left(T^a \cdot T^b\right) = \delta^{ab}/2$ (see (2.56)).

We already mentioned in subsection 3.2.1 that this hypercharge generator differs from the definition of the Standard Model one by a factor

$$Y = \sqrt{\frac{5}{3}}Y_1\ . \tag{3.14}$$

To make this clear, we look again at the subduction $\mathbf{5} \to (\mathbf{3},\mathbf{1})_{-1/3} \oplus \cdots$ claimed above. The representation $(\mathbf{3},\mathbf{1})$ obviously corresponds to the first three components of the $\mathbf{5}$. Y_1 acts on this subspace by multiplication with $\frac{-2}{\sqrt{60}}$, therefore $Y/Y_1 = \frac{-1}{3}\frac{\sqrt{60}}{-2} = \sqrt{\frac{5}{3}}$.

Exotic Gauge Bosons. We have already seen above that the $\mathbf{24}$ of SU(5) contains additional bosons in representations $(\mathbf{3},\bar{\mathbf{2}})_{-5/6} \oplus (\bar{\mathbf{3}},\mathbf{2})_{+5/6}$. They correspond to the generators $E_{a4}^{(1,2)}$ and $E_{a5}^{(1,2)}$ for $1 \le a \le 3$, let's call them $X_a^{(1,2)}$ and $Y_a^{(1,2)}$.

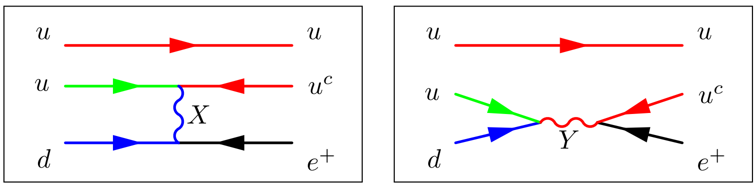
Similar to how we defined $W^\pm = \frac{1}{\sqrt{2}}\left(W^1 \mp \mathrm{i}W^2\right)$, let

$$X_a^\pm = \frac{1}{\sqrt{2}}\left(X_a^{(1)} \pm X_a^{(2)}\right) \quad \text{and} \quad Y_a^\pm = \frac{1}{\sqrt{2}}\left(Y_a^{(1)} \pm Y_a^{(2)}\right)\ . \tag{3.15}$$

$X_a^\pm$ and $Y_a^\pm$ are charge eigenstates with charge $\pm\frac{4}{3}$ and $\pm\frac{1}{3}$, respectively. If we write the $\mathbf{24}$ gauge field as a matrix, it looks like this:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\,\underline{G} - \frac{2B}{\sqrt{30}}\mathbb{1}_{3\times3} & \boldsymbol{X}^- & \boldsymbol{Y}^- \\ (\boldsymbol{X}^+)^T & \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\ (\boldsymbol{Y}^+)^T & W^- & \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{pmatrix}\ . \tag{3.16}$$

Writing down the complete Lagrangian one finds that X and Y mediate additional gauge interactions: There are *leptoquark vertices* coupling leptons and quarks and also *diquark vertices* coupling quarks and antiquarks. Those make proton decay possible, as can be seen for example in the following Feynman diagrams:



(3.17)

Symmetry Breaking. During the process of symmetry breaking, we will find that we have to introduce even more exotic particle content.

First, we have to break the symmetry down from SU(5) to the Standard Model. For this, one introduces a Lorentz-scalar Higgs field in the $\mathbf{24}$ that acquires a vacuum expectation value. 12 of the degrees of freedom are Goldstone bosons which are eaten by the X and Y bosons, those acquire GUT scale masses in the process. The remaining 12 degrees of freedom are massive, this doesn't contradict observations because they do not couple to fermions.

To further break down the symmetry to $\mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{em}}$ we know that we need a $(\mathbf{1},\mathbf{2})_{1/2}$ Higgs doublet. To get it, we include another Higgs field in the fundamental representation $\mathbf{5}$ because

$$\mathbf{5}_H \to (\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{3},\mathbf{1})_{-1/3} \ . \tag{3.18}$$

This results in a new problem: The new Higgs triplet could also mediate proton decay and therefore has to have a GUT scale mass. The parameters have to be incredibly fine-tuned in order to allow the Higgs doublet and the Higgs triplet to have such different masses. Not even the introduction of supersymmetry (which explains the Hierarchy problem in the Standard Model) solves this problem. This is called the *doublet-triplet splitting problem* and persists in a lot of other GUT theories as well. Another similar problem is that we need the $\mathbf{24}$ Higgs and the $\mathbf{5}$ to take vacuum expectation values which differ by more than 10 orders of magnitude.

Note that in the MSSM there are two Higgs doublets (see subsection 3.1.2). They are organized into two SU(5) multiplets $\mathbf{5}$ and $\bar{\mathbf{5}}$ and there are two additional triplets T_u and T_d. Summarizing, this is how all MSSM matter arises from SU(5) representations:

$$\begin{aligned} &\mathbf{10} \to (Q_L, u_R^c, e_R^c), \quad \bar{\mathbf{5}}_M \to (d_R^c, L_L), \quad \mathbf{1} \to \nu_R^c \\ &\mathbf{5}_H \to (T_u, H_u), \quad \bar{\mathbf{5}}_H \to (T_d, H_d) \ , \end{aligned} \tag{3.19}$$

Further Implications. As discussed above, at the GUT scale we have gauge coupling unification, $\sqrt{5/3}g' = g = g_s$. This leads to a prediction of the Weinberg angle (3.1) at the GUT scale:

$$\cos^2 \Theta_W = \frac{g^2}{g^2 + g'^2} = \frac{1}{1+\frac{3}{5}} = \frac{5}{8} \ . \tag{3.20}$$

Also, because of the requirement of SU(5) symmetry, there is less freedom in the Yukawa mass terms for the fermions. This leads to another prodiction: At the GUT scale,

$$m_e = m_d, \quad m_s = m_\mu, \quad m_b = m_\tau \ , \tag{3.21}$$

but since the masses run differently as functions of the energy scale, those relations do not need to hold at our electroweak scale. However, this simple model predicts that even for the running couplings $m_\mu/m_e \approx m_s/m_d$, conflicting strongly with observations. This problem can be fixed by making the Higgs sector more complicated.

3.3 String Theory

Note. Taken from: [12, 20]. Another good review is [21].

3.3.1 Introduction

The (supersymmetric) Standard Model is only an effective theory valid at our low energy scales. We know this because we have to take this fact into consideration whenever we calculate physical quantities in particle physics, by the procedure of regularization and renormalization. Also, the Standard Model does not incorporate gravity, it contains some features that are considered unnatural, and there are several other questions it does not answer.

String theory is a UV completion of Yang-Mills theory in the sense that string theory is consistent and produces Yang-Mills theory in the low-energy limit. An appealing fact is that the particle spectrum of this low-energy limit necessarily includes gravitons, therefore string theory is a unification of gauge theory and gravity.

Quantizing the string action. The way it works is that we write down an action for a *string* propagating through spacetime. Just like the action for a point particle is given by the length of its world line, the action of the string is given by the area of its *world sheet* Σ: We describe a propagating string in D-dimensional spacetime with a parametrization $X^\mu(\sigma^a)$ (for $a \in \{0,1\}$ and $0 \leq \mu < D$), then

$$S = -T \int_\Sigma \mathrm{vol} = -T \int \sqrt{-\det\left(\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b}\right)}\, \mathrm{d}^2\sigma \; . \tag{3.22}$$

The constant T is called the *string tension*. Related quantities are the *Regge slope* $\alpha' = \frac{1}{2\pi T}$ and the *string length* $\ell_s = \sqrt{\alpha'}$. This can be viewed as a theory of D fields X^μ on the world sheet and we can quantize it, generally speaking, by the usual procedure.

Quantization is difficult, however, as we have a high amount of gauge freedom. Like in Yang-Mills theory we deal with it using *BRST quantization*, but it turns out that the BRST algebra becomes anomalous except if $D = 26$. In *superstring theory* where we add fermions as well, the critical dimension changes and becomes $D = 10$. Therefore superstring theory is only consistent in ten spacetime dimensions.

Emergent spacetime. The viewpoint that was just presented is quite basic from a modern point of view. It is more sophisticated to consider general *conformal field theories* on the world sheet. The X theory above is an example for a CFT, other examples are the world sheet fermions introduced in superstring theory and the Faddeev-Popov ghost systems we need for BRST quantization. One way to think about string theory is that we can in principle take an arbitrary CFT on the world sheet and quantize it, BRST quantization works as long as the CFT's total *central charge* is zero. The CFT with 10 bosons, 10 fermions and some Faddeev-Popov ghosts is just one example for a CFT with zero central charge, in principle we could use completely different CFTs as well. In this description, spacetime is an emergent feature, and not necessarily equal to $\mathbb{R}^{1,9}$.

This is a good thing because we want to get $(3+1)$-dimensional spacetime as a low-energy limit from string theory if it is to be a fundamental description of nature. In general, the spacetime structure emerging in this manner can be weird and it might not be possible to describe it in terms of classical geometry. We assume, however, that a large volume limit exists where classical geometry is applicable: In the following we simply talk about strings propagating in a 10-dimensional manifold $\mathcal{M}_{10}$. This is a useful picture, but there are some caveats [22, Ch. 14.3].

String compactification. We will follow [22], see also [20, Ch. 7.6]. In order to describe reality, we take

$$\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{M}_6 \tag{3.23}$$

with $\mathcal{M}_6$ compact. The background metric G of the total space $\mathcal{M}_{10}$ needs to factor into a Minkowski metric η on $\mathbb{R}^{1,3}$ and some other metric g on $\mathcal{M}_6$:

$$G_{MN}(x,y) = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{mn}(y) \end{pmatrix} \; . \tag{3.24}$$

A necessary condition for unbroken supersymmetry is the existence of a covariantly constant spinor ϵ on $\mathcal{M}_6$,

$$\nabla \epsilon = 0 \tag{3.25}$$

(where ∇ is the background covariant derivative on spinors). One shows that this implies $R_{MN}\Gamma^N \epsilon = (R_{\mu\nu}\Gamma^\nu \oplus R_{mn}\Gamma^n)\,\epsilon = 0$ (where Γ are Gamma matrices and R the Ricci tensors). Since $\mathbb{R}^{1,3}$ is flat we need $R_{mn}\Gamma^n\epsilon = 0$ which is only possible for non-zero ϵ if

$$R_{mn} = 0 \; . \tag{3.26}$$

Thus, our necessary condition is that the internal manifold $\mathcal{M}_6$ admits a Ricci-flat metric.

As discussed in subsection 2.1.4, Calabi-Yau manifolds satisfy this necessary condition. But not every Ricci-flat compact Kähler mainfold is Calabi-Yau (see subsection B.4.4 in the appendix for more details). It turns out, however, that $\mathcal{M}_6$ being Calabi-Yau according to definition B.94 is a sufficient condition for having exactly the same amount of supersymmetry in the 4-dimensional theory as in the original theory before compactification ($\mathcal{N}_4 = \mathcal{N}_{10}$).

The string landscape. If we want to define a theory in order to arrive at a low-energy limit, specifying the internal manifold $\mathcal{M}_6$ is not enough. We have to generalize compactifications and also allow *background fluxes*, this is called flux compactification [22, Ch. 17.3] and will be crucial for us later on. Background fluxes are non-zero vacuum expectation values of the field strengths of certain form fields that appear in the particle spectrum (see below).

This leads to a huge number of possible compactifications (called *flux vacua*) of the so far almost[5] unique string theory in 10 dimensions. Each of those flux vacua has a different resulting particle spectrum in 4 dimensions, the set of all those vacua is called the *string landscape*. To date, it is not known which – if any – compactification is realized in nature. A commonly quoted ballpark figure for the amount of possible flux compactifications is 10^{500} [23].

3.3.2 Type IIB Superstring Theory

Let us go back to superstring theory before compactification. After quantization we find a spectrum of vibrational states of the superstring, having different masses and transformation behaviors under $\mathrm{SO}(1,9)$. We are not interested in those states with $m^2 > 0$ because they are not relevant for the low-energy limit.

Open string spectrum: Consider the open string with free endpoints (NN boundary conditions). There is still a choice to be made about the boundary conditions of the fermionic CFT: The Ramond (R) sector contains strings with periodic boundary conditions while the Neveu-Schwarz (NS) sector contains strings with anti-periodic boundary conditions. The spectrum can be summarized as follows [20]:

Sector	Statistics	Little Group	Representation	m^2 (units of $2\pi T$)
NS_-	boson	SO(9)	$\mathbf{1}$	$-1/2$
NS_+	boson	SO(8)	$\mathbf{8_v}$	0
R_-	fermion	SO(8)	$\mathbf{8_c}$	0
R_+	fermion	SO(8)	$\mathbf{8_s}$	0

The index $\pm$ denotes *G-parity*, see [20]. $\mathbf{8_s}$ and $\mathbf{8_c}$ are Weyl spinors with positive and negative chirality, respectively. Note that this still contains an unphysical tachyon.

Closed string spectrum: A closed string excitation can be described as the tensor product of a left-moving and a right-moving state, both of which can be independently $\mathrm{NS}_\pm$ or $\mathrm{R}_\pm$ (but not all pairings are possible). We find the following spectrum (up to interchange of left- and right-movers) [20]:

Sector	Statistics	Little Group	Representation	m^2 (units of $2\pi T$)
$(\mathrm{NS}_-,\mathrm{NS}_-)$	boson	SO(9)	$\mathbf{1}$	-2
$(\mathrm{NS}_+,\mathrm{NS}_+)$	boson	SO(8)	$\mathbf{8_v}\otimes\mathbf{8_v}$	0
$(\mathrm{R}_-,\mathrm{R}_-)$	boson	SO(8)	$\mathbf{8_c}\otimes\mathbf{8_c}$	0
$(\mathrm{R}_+,\mathrm{R}_+)$	boson	SO(8)	$\mathbf{8_s}\otimes\mathbf{8_s}$	0
$(\mathrm{R}_-,\mathrm{R}_+)$	boson	SO(8)	$\mathbf{8_c}\otimes\mathbf{8_s}$	0
$(\mathrm{NS}_+,\mathrm{R}_+)$	fermion	SO(8)	$\mathbf{8_v}\otimes\mathbf{8_s}$	0
$(\mathrm{NS}_+,\mathrm{R}_-)$	fermion	SO(8)	$\mathbf{8_v}\otimes\mathbf{8_c}$	0

It is interesting how those product representations decompose under the group action (see also subsection 2.3.4). For example, the $(\mathrm{NS}_+,\mathrm{NS}_+)$ boson decomposes as

$$\mathbf{8_v}\otimes\mathbf{8_v} = \underbrace{\mathbf{1}}_{\text{Dilaton }\Phi} \oplus \underbrace{\mathbf{28}}_{B_{\mu\nu}\text{ (antisym.)}} \oplus \underbrace{\mathbf{35_v}}_{\text{Graviton }G_{\mu\nu}\text{ (sym. traceless)}} . \tag{3.27}$$

In the (R,R)-sector, we get p-form fields C_p and in the mixed sector we get spacetime fermions, dilatinos and gravitinos.

[5] We will see later that there are five consistent superstring theories. They are connected through certain dualities.

Digression: String coupling, gravity etc. Now that we have found the particle spectrum, we need to go back to the action (3.22) and rewrite it including our new fields. For example, if we want to describe strings propagating in a curved background, we should replace the appearing Minkowski metric $\eta_{\mu\nu}$ with $G_{\mu\nu}$. We also add for example a term

$$\sim \int_\Sigma R\,\Phi(X)\ \mathrm{vol} \sim \chi(\Sigma)\,\Phi \tag{3.28}$$

where R is the Ricci scalar on the world sheet, Φ the dilaton and $\chi(\Sigma)$ the Euler characteristic (for a rigorous treatment see [20]). Putting this into the path integral shows that the string coupling strength g_s is dynamical and

$$g_s = \mathrm{e}^{\Phi}\ . \tag{3.29}$$

The graviton appearing in the closed string spectrum is the justification for the claim made above that string theory contains general relativity in the low-energy limit. It is interesting that we can also derive the Einstein equation for it: As mentioned before, string theory is a conformal field theory and therefore scale invariant. Hence, its beta functions must all be zero. Explicitly calculating the beta function for the background metric and setting it to zero yields exactly the Einstein equation to first order, and therefore string theory reduces to GR in the low-energy limit. Fascinatingly, we are forced to a fluctuating dynamical background even if we started out from a fixed metric.

The GSO projection. In order to get a consistent theory, we cannot simply combine all the sectors listed above – we can only keep some of them. This is called the *GSO projection*. There are only two consistent[6] theories of closed strings. They are called type II theories because they have $\mathcal{N} = 2$ supersymmetry:

- In *type IIB theory* we keep the sectors $(\mathrm{NS}_+, \mathrm{NS}_+)$, $(\mathrm{R}_+, \mathrm{R}_+)$, $(\mathrm{NS}_+, \mathrm{R}_+)$ and $(\mathrm{R}_+, \mathrm{NS}_+)$.
- In *type IIA theory* we keep the sectors $(\mathrm{NS}_+, \mathrm{NS}_+)$, $(\mathrm{R}_+, \mathrm{R}_-)$, $(\mathrm{NS}_+, \mathrm{R}_-)$ and $(\mathrm{R}_+, \mathrm{NS}_+)$.

We just mention that there are three more consistent string theories:

- *Type I theory* consists of open strings. We can construct it from an orientifold projection of IIB, but then we still have to add the open string NS_+ and R_+ sectors and also 32 spacetime-filling D9-branes. This has $\mathcal{N} = 1$ supersymmetry.
- In *heterotic string theory* one combines left-moving superstrings with right-moving bosonic strings compactified on 16 of their 26 dimensions. There are two types, the SO(32) heterotic string and the $E_8 \times E_8$ heterotic string.

From now on we will be mainly concerned with type IIB theory.

Type IIB supergravity action. All in all we have the following bosonic states in type IIB:

- The scalar dilaton Φ and the graviton $G_{\mu\nu}$ which were already discussed above.
- The 2-form field $B_{\mu\nu}$ called the *Kalb-Ramond field*. The corresponding field strength is named

$$H_3 = \mathrm{d}B_2\ . \tag{3.30}$$

- C_p form fields for $p \in \{0, 2, 4\}$ with field strengths usually defined as

$$F_{p+1} = \mathrm{d}C_p - C_{p-2} \wedge H_3 \qquad (F_1 = \mathrm{d}C_0) \tag{3.31}$$

  where C_4 is self-dual in the sense that $F_5 = *F_5$ (this is a constraint which has to be implemented at the level of the equations of motion).

  The Bianchi identity for those field strengths reads $\mathrm{d}F_{p+1} = H_3 \wedge F_{p-1}$ $(\mathrm{d}F_1 = 0)$.

[6] Meaning that they don't have monodromies, don't contain tachyons and are modular invariant.

The string frame type IIB low-energy effective action is given by [24]:

$$
\begin{aligned}
S_{IIB} &= S_{\mathrm{NS}} + S_{\mathrm{R}} + S_{\mathrm{CS}} \\
S_{\mathrm{NS}} &= \frac{1}{2\kappa_{10}^2}\left\{\int \mathrm{d}^{10}x\sqrt{-G}\,\mathrm{e}^{-2\Phi}R + \int \mathrm{e}^{-2\Phi}\left(4\,\mathrm{d}\Phi\wedge *\mathrm{d}\Phi - \frac{1}{2}H_3\wedge *H_3\right)\right\} \\
S_{\mathrm{R}} &= -\frac{1}{4\kappa_{10}^2}\int\left(F_1\wedge *F_1 + F_3\wedge *F_3 + \frac{1}{2}F_5\wedge *F_5\right) \\
S_{\mathrm{CS}} &= -\frac{1}{4\kappa_{10}^2}\int\left(C_4 - \frac{1}{2}B_2\wedge C_2\right)\wedge H_3\wedge F_3\ .
\end{aligned}
\tag{3.32}
$$

We have split the action in one part containing only NS-NS fields, one part containing only R-R fields and the Chern-Simons interaction part S_{CS}. κ_{10}^2 is the physical gravitational coupling and $\frac{1}{2\kappa_{10}^2} = \frac{2\pi}{\ell_s^8}$.

There is a number of different ways of writing this action. Quite often (e.g. [20, 25]) people use $\tilde{C}_4 = C_4 - \frac{1}{2}B_2\wedge C_2$ instead of C_4, this simplifies the Chern-Simons term. Expressing F_5 in those new variables gives

$$
F_5 = \mathrm{d}\tilde{C}_4 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge \mathrm{d}C_2\ . \tag{3.33}
$$

Democratic Formulation. Yet another way is the so-called democratic formulation [26, 27]. It uses C_p form fields for $p\in\{0,2,4,6,8\}$ with field strengths as in (3.31), imposing the constraint

$$
F_p = (-1)^{(p-1)/2} *F_{10-p} \tag{3.34}
$$

on the level of the equations of motion. The complete action is

$$
S_{IIB} = \frac{1}{2\kappa_{10}^2}\int \mathrm{e}^{-2\Phi}\left(R\ \mathrm{vol} + 4\,\mathrm{d}\Phi\wedge *\mathrm{d}\Phi\right) - \frac{1}{4\kappa_{10}^2}\int\left\{\mathrm{e}^{-2\Phi}\left(H_3\wedge *H_3\right) + \frac{1}{2}\sum_{p=0}^{4} F_{2p+1}\wedge *F_{2p+1}\right\}\ . \tag{3.35}
$$

Note that the Chern-Simons term is absent. But, after imposing the constraints and eliminating C_6 and C_8, this action leads to the same equations of motion like the action in (3.32). For an overview, see [24].

3.3.3 Dp-Branes

Once more we go back to bosonic strings propagating in flat spacetime, those strings can be open or closed. For each end of an open string, for each spacetime dimension, there are two possible boundary conditions: We can have either *Neumann boundary conditions* where $\partial_{\sigma^1}X^\mu = 0$ at the end point, this means that no momentum can flow off the string in that direction. The other option is $\partial_{\sigma^0}X^\mu = 0$ meaning that the string end point is fixed in this direction.

We get the picture that each end of the string can move freely in some directions and is fixed in the others. More generally, the end of the string can be confined to a hypersurface. We define: A Dp-brane is a $(p+1)$-dimensional hypersurface on which open strings can end.

When we quantize the open string with Dp-branes present, we make the following observations: On strings that begin and end on the same brane, parallel and normal excitations transform differently from the perspective of the brane (under $\mathrm{SO}(p,1)$). The parallel excitations form a massless vector, transforming in the fundamental of $\mathrm{SO}(p-1)$, showing that the Dp-brane hosts a $\mathrm{U}(1)$ gauge theory. If we have a “stack of N branes”, the end points of the strings have to be marked with “Chan-Paton factors” indicating to which brane they belong. This gives a $\mathrm{U}(N)$ gauge theory. (This can be made mathematically rigorous using coherent sheaf models.)

More complicated setups like several intersecting stacks of branes can be very interesting. There are strings propagating along one of the brane stacks, they correspond to the gauge bosons of that stack’s gauge group. Strings stretching between such stacks are fermions transforming in bifundamental representations of the product group. Because of the string tension, they are confined to the intersection of the brane stacks. With these ingredients we can model something similar to the Standard Model [20, Ch. 7.5]. Several generations of fermions can be modeled easily be having a more complicated geometry in which there are several intersection points. Gravity is mediated by closed strings, they are not bound to the branes.

Branes as Dynamical Objects. In subsection 3.3.1 we found that the terms in the action containing the dilaton and the graviton make the string coupling and the background geometry into dynamical objects. Something similar happens for branes: Through their coupling to the different kinds of strings, a Dp-brane can not be static, but it is fluctuating. The coupling consists of two terms:

- The *Dirac-Born-Infeld action* couples the brane to the NS-NS sector. It is a generalization of (3.22) where we now integrate over the world volume of the Dp-brane: Let $\iota : D_p \to \mathcal{M}_{10}$ be the embedding of the brane into spacetime, then

$$S_{\mathrm{DBI}} = -\frac{2\pi}{\ell_s^{p+1}} \int_{D_p} \mathrm{e}^{-\iota^*\Phi} \sqrt{-\det\left(\iota^* G + 2\pi l_s^2 \mathcal{F}\right)}\, \mathrm{d}^{p+1}\xi \ . \tag{3.36}$$

  Here $\mathcal{F} = F + T\,\iota^* B$, F is the field strength of the gauge theory over the brane and B the Kalb-Ramond field.

- The *Chern-Simons action* expresses that D-branes are sources of the R-R C_p form fields: To lowest order it is

$$S_{\mathrm{CS}} = -\frac{2\pi}{\ell_s^{p+1}} \int_{D_p} \iota^* C_{p+1} \ , \tag{3.37}$$

  we will write it down completely in (5.22).

  Note the similarity to electrodynamics: Let A be the electromagnetic four potential and $F = \mathrm{d}A$. If there is a current along a 1-dimensional current string $J \in Z_1$ (definition A.41), the action contains one term $\sim \int F \wedge *F$ (like in (3.35)) and a term $\sim \int_J A$ like here.

  If the action of a brane does not have a Chern-Simons term, the brane is uncharged and dynamically unstable. This means a theory can only contain Dp-branes if there are C_{p+1} form fields. The following table (taken from [21]) summarized which branes are contained in which theory:

Theory	p
Type I	$1, 5, 9$ (strings moving freely)
Type IIA	$0, 2, 4, 6, 8$
Type IIB	-1 (Instanton), $1, 3, 5, 7$

String Theory. We see that string theory has much more content than what we initially thought. The fundamental string (F1-string or F-string) is just one part of the spectrum of string theory, we also have:

- The D-branes that were discussed above, among them the D-string in type IIB theory. The D-string and the F-string are fundamentally different: An F-string ends on D-branes, but not vice versa.

  Further, the coupling of the D-branes is $\sim \mathrm{e}^{-\Phi} = g_s^{-1}$. This shows that D-branes are non-perturbative objects in string theory. They are also called BPS-states or the BPS spectrum of the theory.

- The F-string sources the Kalb-Ramond field B_2 (just like the D-string sources C_2). There is a 6-form dual to B_2 (with $H_7 = *H_3 = *\mathrm{d}B_2$), it couples to a 5-brane. This is, however, not a D-brane (meaning that open strings do not end there) – it is called the NS5-brane. Its coupling goes $\sim g_s^{-2}$ [21].

- Later we will encounter more exotic objects like $\binom{p}{q}$-strings ending on $[p,q]$-branes [5, Ch. 2.1].

p-form Fields with Source Terms. We have seen above that Dp-branes are sources of C_{p+1} form fields. It will be important to understand how such source terms influence the behavior of the fields. As mentioned above, we have the same situation in electrodynamics. We will now have a quick look at electro- and magnetostatics in order to develop an intuition.

The action of electrodynamics is $S = -\frac{1}{4}\int F\wedge *F - \int A\wedge *J$. When we consider static configurations, this splits up into an electric and a magnetic part. The electrostatics action for a point source $\rho = Q\,\delta^{(3)}(\boldsymbol{x})\ \mathrm{vol} \in \Omega^3$ is

$$\frac{1}{4}\int E\wedge *E - \int_\rho \Phi \tag{3.38}$$

where $E = \mathrm{d}\Phi$, this is like a 0-form coupling to a D(-1)-brane in 3 dimensions. Note that we have identified $\rho \in \Omega^3$ with its Poincaré dual $\rho = Q\,\{x = y = z = 0\} \in Z_0$.

The equations of motion derived from this action are $\mathrm{d}{*}\mathrm{d}\Phi = \rho$, equivalent to $\Delta\Phi = Q\,\delta^{(3)}(\boldsymbol{x})$. The well-known solution is

$$\Phi = \frac{Q}{4\pi}\frac{1}{r}\ . \tag{3.39}$$

Consider now a static electric current, e.g. $*j = J\,\delta(x)\delta(y)\ \mathrm{d}z$. The magnetic part of the action is

$$-\frac{1}{4}\int B\wedge *B + \int_j A \tag{3.40}$$

where $B = \mathrm{d}A$ and we have again identified j with its Poincaré dual. This is like a 1-form coupling to a D0-brane in 3 dimensions.

The equations of motion here are similarly $\mathrm{d}{*}\mathrm{d}A = j$. Since this is something like a Poisson equation in the x and y directions only, we expect a logarithmic behavior. What makes this a bit more tricky is that A is a 1-form and $\mathrm{d}{*}\mathrm{d} \neq \Delta$. The situation becomes easier if we work with the dual field $\mathrm{d}C_0 = F_1 = *B$. Integrating over a circle in the x-y-plane, we see

$$J = \int_{B^1} \mathrm{d}{*}\mathrm{d}A = \oint_{S^1} *\mathrm{d}A = \oint_{S^1} \mathrm{d}C_0 \tag{3.41}$$

with the known solution

$$C_0 = \frac{J}{2\pi}\arctan\frac{y}{x} = \frac{J}{2\pi}\Im\left(\ln(x+\mathrm{i}y)\right) \tag{3.42}$$

(up to terms regular in $x + \mathrm{i}y$).

What will be important for us is the following:

- When we have a D$(n-3)$-brane in n dimensions, the situation will always be as in (3.42): The coupling $\int_{\mathrm{D}(n-3)} C_{n-2}$ leads to an equation of motion $\mathrm{d}{*}\mathrm{d}C_{n-2} = \delta(x)\delta(y)$ (where x and y are the normal directions). Integrating this gives $\oint_{S^1}\mathrm{d}C_0 = 1$, solved by (3.42). This solution can not be globally defined.
- If we have a Dp-brane with $p < (n-3)$ we will always get a solution like (3.39), without a branch cut.

3.3.4 Dualities

The five consistent string theories we have been talking about so far are actually all linked through different dualities. A duality between weakly and strongly coupled theories, perturbative in ℓ_s but not in g_s, is called an *S-duality.* If the situation is reversed and the duality is perturbative in g_s, it is called a *T-duality.* The case where the duality is perturbative neither in ℓ_s nor in g_s is called *U-duality.*

S-Duality. Proving S-duality is difficult, but after our previous discussion we can already understand S-duality in type IIB theory. It interchanges g_s with g_s^{-1} and simultaneously swaps the F-string with the D-string and the NS5-brane with the D5-brane. Another example for S-duality exists between type I theory and the SO(32) heterotic string. Here, the D-string of type I is mapped to the fundamental heterotic string. [21]

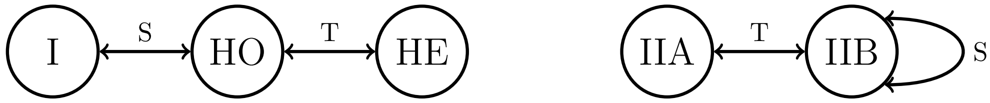


Figure 3.2: Dualities among the different superstring theories. [28]

T-Duality. T-duality can already be understood when considering the most simple type of compactification: So-called *Kaluza-Klein compactification* where one of the dimensions is replaced with a circle S^1. Already in ordinary point particle theory, applying this compactification to e.g. a massless scalar gives a tower of massive states with mass squares

$$m_n^2 = \frac{n^2}{R^2} \tag{3.43}$$

where R is the radius of the S^1.

When we consider K-K compactification of a closed bosonic string, something new happens: The string can be wrapped around the S^1 with winding number ω. The previously massless excitations of the string acquire a mass, depending on n and ω:

$$m_{n\omega}^2 = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'} \tag{3.44}$$

This is dual under $n \leftrightarrow \omega$ and $R \leftrightarrow R' = \frac{\alpha'}{R}$. It is also important to know that the string coupling transforms as [22, Ch. 14.2]

$$g_s' = \frac{\ell_s}{R}\, g_s \; . \tag{3.45}$$

When doing this in superstring theory, we have to carefully analyze the chiralities of the spinors. The result is that type IIB theory compactified on an S^1 with radius R is equivalent to type IIA theory compactified on an S^1 with radius R'. Dp-branes are mapped to D$(p \pm 1)$-branes, depending on whether the S^1 is in the direction of the brane $(-)$ or perpendicular to it $(+)$.

There is still T-duality if we perform Calabi-Yau compactifications instead of these simple compactifications on S^1, it then is called “mirror symmetry”. T-duality also exists between the SO(32) and the $E_8 \times E_8$ heterotic theories. All dualities encountered so far are summed up in figure 3.2, we are going to talk more about dualities in section 4.2.

Chapter 4

F-Theory

Note. In this chapter we will mainly follow [5, 28, 29] and the end of [22]. Also, [30] provides a good introduction to elliptic fibrations.

4.1 Introduction

We now want to start to work in type II superstring theory with D-branes. A problem we haven't mentioned before is that type II compactifications with D-branes are often inconsistent or unstable, because the branes carry charge as mentioned in subsection 3.3.3 (see also [31, Ch. 2.2]). From the point of view of string theory, absence of charges is equivalent to tadpole cancellation conditions, which guarantee that there are no anomalies.

The solution is to perform an *orientifold compactification* instead. Orientifolds are a generalization of orbifolds, quotients $\mathcal{M}/G$ of manifolds under a group action (see subsection C.2.3). In an orientifold additional states are projected out according to their world sheet parity ($\mathcal{M}/(G_1 \cap \Omega G_2)$ where Ω is the world sheet parity operator). More specifically, we want to act with

$$\mathbb{Z}_{2,O} = \mathbb{Z}_{2,g}\, P\, (-1)^{F_L} \tag{4.1}$$

on the manifold. $\mathbb{Z}_{2,g}$ is a holomorphic involution of the Calabi-Yau,

$$P : \sigma^1 \to (2\pi - \sigma^1) \tag{4.2}$$

is the parity transformation of the world sheet and F_L the left-moving fermion number. The first advantage of this is that it removes half of the supersymmetry generators such that we get an $\mathcal{N} = 1$ theory.

Furthermore, the action of $\mathbb{Z}_{2,g}$ will typically leave certain subvarieties fixed, those become the location of *O-planes*. Such O-planes carry negative charge, hence we can build consistent models containing D-branes. More precisely, an Op-plane has charge [22, Ch. 10.6]

$$Q_{\mathrm{O}_p} = -2^{p-4}\, Q_{\mathrm{D}_p} \ , \tag{4.3}$$

hence we need for example four[1] D7-branes for each O7-plane to exactly cancel the charges. Note however that the location of an O-plane is fixed, they do not carry dynamical degrees of freedom.

From now on we will concern ourselves with orientifold compactifications of type IIB superstring theory containing D7-/D3-branes and O7-/O3-planes, those give consistent configurations.

[1] Because $Q_{\mathrm{O}_7} = -8\, Q_{\mathrm{D}_7}$ one would expect eight D7-branes, but equation (4.3) counts the charge in the double cover Calabi-Yau. After orbifolding, we count half as many branes with twice the charge.

$\mathrm{SL}(2,\mathbb{Z})$ **Invariance.** We discussed in subsection 3.3.3 what the C_0 field looks like close to a D7-brane: $C_0 \sim \Im \ln(z)$ (where $z = x + \mathrm{i}y$ and x, y are the directions normal to the brane). Let us now define the *axio-dilaton*

$$\tau = C_0 + \frac{\mathrm{i}}{g_s} \,, \tag{4.4}$$

due to constraints from supersymmetry it must be a holomorphic function in z. Therefore,

$$\tau(z) = \tau_0 + \frac{1}{2\pi\mathrm{i}} \ln(z - z_0) + (\text{regular terms}) \tag{4.5}$$

if the D7-brane is located at $z = z_0$. We see that the brane introduces a *monodromy*, meaning that if we go in a circle around it,

$$\tau \to \tau + 1 \,. \tag{4.6}$$

This seems like an inconsistency of the theory at first, but actually the theory is invariant under $\tau \to \tau+1$ (if the other fields also transform accordingly, see below).

The solution (4.5) also shows us that the backreaction of D7-branes on the geometry is strong: In their presence, there will always be areas where g_s is very small as well as areas where g_s is very large (since $g_s^{-1} \sim -\ln|\frac{z-z_0}{\lambda}|$ for some λ) so that perturbation theory is not applicable. On the other hand, we know from subsection 3.3.4 that type IIB theory has invariance under S-duality where

$$\tau \to -\tau^{-1} \,. \tag{4.7}$$

Equations (4.6) and (4.7) are only two examples of a more general symmetry of type IIB string theory: It is invariant under the $\mathrm{SL}(2,\mathbb{Z})$-symmetry

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \qquad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \,. \tag{4.8}$$

The fields F_5 and G are unchanged. (On the classical level, the action is even invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations, but this symmetry is reduced to $\mathrm{SL}(2,\mathbb{Z})$ through quantum effects.) Going around a D7-brane corresponds to the action of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and S-duality corresponds to $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Incidentally, those two elements already generate the whole group $\mathrm{SL}(2,\mathbb{Z})$. Going around an O7-plane corresponds to the element $-T^{-4}$.

We finally notice that this symmetry exchanges the C_2 and B_2 fields which couple to the D-string and the F-string, respectively. Therefore we should also combine the D-string and the F-string to an $\mathrm{SL}(2,\mathbb{Z})$ multiplet: The $\binom{p}{q}$-string which has p units of B_2-charge and q units of C_2-charge. Then, we need to introduce $[p, q]$-branes as the object on which $\binom{p}{q}$-strings can end. The monodromy induced by a $[p, q]$-brane is $\begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix}$. Note that locally, using an $\mathrm{SL}(2,\mathbb{Z})$ transformation, every brane can be made to look like a D7-brane (which is a $[1, 0]$-brane).

Geometric Description. The idea of F-theory [4] is to make these invariances an intrinsic part of the theory. The axio-dilaton τ is interpreted as the complex structure modulus of a complex torus T^2 which has the same $\mathrm{SL}(2,\mathbb{Z})$ symmetry, as discussed in subsection 2.1.1. Since τ varies over the complex 3-dimensional internal manifold B_3, we get a smoothly varying torus over B_3, in other words an *elliptic fibration*

$$T^2 \to Y_4 \to B_3 \tag{4.9}$$

(see section 4.3). The two extra dimensions introduced here are not physical though, they just provide a bookkeeping tool for the $\mathrm{SL}(2,\mathbb{Z})$ symmetry. F_3 and H_3 can be interpreted as components of a 12-dimensional 4-form $\hat{F}_4$.

The *F-theory conjecture* states that the physics of a type IIB orientifold compactified on B_n is encoded in the geometry of the elliptically fibered Y_{n+1}. We will later see that Y_{n+1} has to be Calabi-Yau, and that the singularities of it encode the locations of the D7-branes and O7-planes, and the gauge groups living there. In this way, F-theory gives a *non-perturbative* description of type IIB string theory.

As a summary, F-theory is the natural language to talk about how type IIB backgrounds with varying coupling are patched together in the presence of D-branes. Because of the corresponding $\mathrm{SL}(2, \mathbb{Z})$ invariances of IIB theory and of the torus, the fibrations over the different patches can always be put together. More explicitly, the monodromy (4.6) we have encountered above is not a problem any more because the torus does not change as we encircle the brane. (4.6) only tells us that we can't describe the whole fibration with a single chart.

4.2 F/M-Theory Duality

M-Theory. The strong coupling limit of IIB theory is again IIB theory because of S-duality. What is the strong coupling limit of IIA theory?

The spectrum of type IIA theory contains bound states of D0-branes with mass

$$m_n = \frac{n}{g_s \ell_s} \tag{4.10}$$

for every $n \in \mathbb{N}_+$ [22, Ch. 18.7]. This looks like the Kaluza-Klein spectrum of an 11-dimensional theory which was compactified on a circle of radius

$$R_{11} = g_s \ell_s \ . \tag{4.11}$$

Hence, the low-energy effective action of the strong coupling limit seems to be the *unique supergravity theory in* 11 *dimensions*. The field content of 11D SUGRA is one graviton, one gravitino, and a three-form field A_3. We can perform a dimensional reduction to 10 dimensions and see explicitly that this leads to the field content and action of type IIA SUGRA. For example, the A_3 field encapsulates both the C_3 and the B_2 fields of type IIA via

$$A_3 = C_3 + B_2 \wedge \mathrm{d}x^{10} \ . \tag{4.12}$$

As we have seen, the compactification radius is inversely proportional to the string coupling g_s. This makes sense: at weak coupling, the eleventh dimension is very small and perturbative type IIA theory is valid. At strong coupling, the appropriate description is 11D supergravity.

Up to now, we have only been talking about the low-energy effective field theories. Since string theory is not just its low-energy limit, there must exist an 11-dimensional quantum theory with 11D SUGRA as its low-energy limit . This theory can not be a string theory, it is so far unknown. It has been dubbed *M-theory*.

Moduli of the Torus. We want to compactify M-theory on a torus $T^2 = S^1_A \times S^1_B$. Let us remember some basic facts about T^2: We can write a metric down as

$$\mathrm{d}s^2_{T^2} = \frac{v}{\tau_2} \left[(\mathrm{d}x + \tau_1 \, \mathrm{d}y)^2 + (\tau_2 \, \mathrm{d}y)^2 \right] \ , \tag{4.13}$$

where x and y are the coordinates along S^1_A and S^1_B, respectively. The parameter v is the Kähler modulus of the torus, that is its volume $v = \iint_{[0,1]^2} \sqrt{g} \ \mathrm{d}x\mathrm{d}y$. $\tau = \tau_1 + \mathrm{i}\tau_2$ is the complex structure modulus we already discussed above. From similar integrations, we easily see that $R_A = \int_0^1 \mathrm{d}s|_{\mathrm{d}y=0} = \sqrt{v/\tau_2}$ and analogously $R_B = |\tau| \, R_A$. In the special case of a rectangular torus ($\tau_1 = 0$), we get

$$\tau = \mathrm{i}\frac{R_B}{R_A} \qquad \text{and} \qquad v = R_A R_B \ . \tag{4.14}$$

Duality. Let us now consider M-theory on

$$\mathbb{R}^{1,8} \times \underbrace{S^1_A \times S^1_B}_{T^2} \ . \tag{4.15}$$

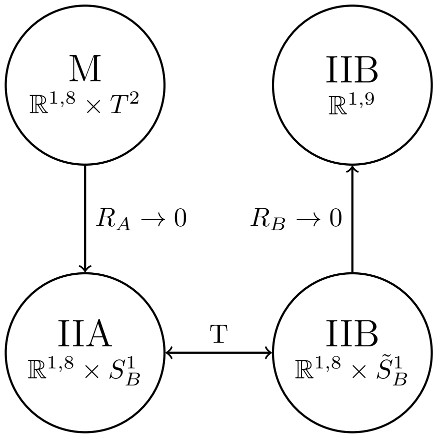


Figure 4.1: Graphical depiction of the limit of vanishing torus volume of M-theory.

If we let one of the radii, R_A say, go to zero, we know that we will get type IIA theory on $\mathbb{R}^{1,8} \times S^1_B$. After that we perform T-duality in the direction of S^1_B, the result is type IIB theory on $\mathbb{R}^{1,8} \times \tilde{S}^1_B$ with radius $\tilde{R}_B = \frac{\ell_s^2}{R_B}$. The limit of $R_B \to 0$ corresponds to $\tilde{R}_B \to \infty$, therefore we get IIB theory on a flat Minkowski spacetime $\mathbb{R}^{1,9}$ in this limit. This procedure is depicted in figure 4.1; for some more detail, see [22, Ch. 18.7] or [5].

We are taking the limit of R_A and R_B going to zero at the same time. According to (4.14), this means taking the limit of vanishing torus volume of M-theory. Crucially, the complex structure modulus is found to become the axio-dilaton of type IIB theory, on the other hand.

To see this, remember from (4.11) that the string coupling of the IIA theory is given by $g_{\text{IIA}} = \frac{R_A}{\ell_s}$. According to (3.45) and (4.14), the coupling of the IIB theory after performing T-duality is then

$$g_{\text{IIB}} = \frac{\ell_s}{R_B}\, g_{\text{IIA}} = \frac{R_A}{R_B} = \Im(\tau)^{-1} \,, \tag{4.16}$$

that is what we needed to show. (The discussion can be generalized to non-rectangular tori [5].)

In order to see what happens with the real part of τ, we write out the M-theory metric in the form

$$\mathrm{d}s^2_M = \mathrm{d}s^2_{T^2} + \mathrm{d}s^2_{1,8} = \frac{v}{\tau_2}\,(\mathrm{d}x + C_1)^2 + \mathrm{e}^{-\frac{2\chi}{3}}\mathrm{d}s^2_{IIA} \,. \tag{4.17}$$

The IIA one-form field C_1 is identified with $\tau_1\,\mathrm{d}y$. Calculation shows that after T-duality, $C_0 = (C_1)_y = \tau_1$, quod erat demonstrandum. (More details: [28].)

F-theory can be understood as being dual to M-theory on $\mathbb{R}^{1,8} \times T^2$, in the limit of vanishing torus volume. This completes our net of dualities (figure 4.2), we will not elaborate on the duality between F-theory and heterotic theories. We have been only talking about direct products $\mathbb{R}^{1,8} \times T^2$, but the construction generalizes to elliptic fibrations as well. Precisely, we are interested in M-theory on a space of the structure

$$\left(T^2 \to Y_4 \to B_3\right) \times \mathbb{R}^{1,2} \,, \tag{4.18}$$

where Y_4 needs to be Calabi-Yau for $\mathcal{N} = 1$ SUSY in the low-energy effective theory [5]. This finally explains why the F-theory elliptic fibration is Calabi-Yau.

4.3 Elliptic Fibrations

Elliptic Curves. As we mentioned above, a fiber bundle where every fiber is a torus T^2 is called an *elliptic fibration*. The reason is that a torus is the same thing as an *elliptic curve*: It is a complex-one dimensional flat space, in other words a Calabi-Yau 1-fold.

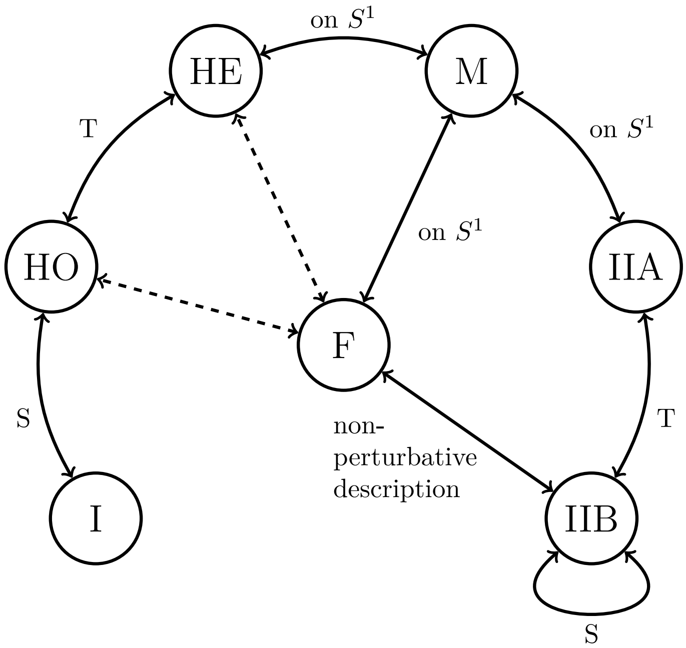


Figure 4.2: Dualities among the superstring theories, F- and M-theory. [28]

The most convenient way to describe an elliptic curve is as a complete intersection in a weighted projective space. Consider $\mathbb{P}^2_{(2:3:1)}$ (see subsection 2.2.2). According to (2.46), a flat hypersurface is given as the zero locus of a polynomial with scaling degree 6. One can show that such a polynomial can always be brought into the so-called *Weierstrass form*,

$$P_W = y^2 - x^3 - f\,xz^4 - g\,z^6 = 0 \tag{4.19}$$

for some $f, g \in \mathbb{C}$ unique up to scaling $(f, g) \sim (\lambda^4 f, \lambda^6 g)$ [32, Ch. II.2].

Going back to equation (4.5) we see that the axio-dilaton and therefore the complex structure modulus of the F-theory torus diverges at the location of the D7-branes. This leads us to the question of when an elliptic curve described by (4.19) is degenerate. For that to happen P_W and $\mathrm{d}P_W$ need to be zero at the same time. An easy calculation (see e.g. [5] or [28]) proves the following simple criterion: An elliptic curve is degenerate if and only if the *discriminant*

$$\Delta = 27\,g^2 + 4\,f^3 \tag{4.20}$$

vanishes.

The relation between the structure of the torus and the parameters f and g of the elliptic curve can be made more explicit: A classical mathematical result states [33]

$$j(\tau) = \frac{4(24\,f)^3}{\Delta} \tag{4.21}$$

where $j : H/\operatorname{PSL}(2,\mathbb{Z}) \to \mathbb{C}$ is *Klein's j-invariant.*

Elliptic Fibrations. We obtain an elliptic fibration if we now let f and g be functions of the coordinates of the base B_n. This is called a *Weierstrass model* of the elliptic fibration (or an E_8-fibration) and – crucially – every elliptic fibration with a section can be represented by a Weierstrass model [32, Ch. II.5]. To be more specific, f and g will be sections of $\mathcal{L}^4$ and $\mathcal{L}^6$, respectively, for some suitable line bundle $\mathcal{L}$ over B_n. Note that then Δ is a section of $\mathcal{L}^{12}$. Next, we will figure out which bundle $\mathcal{L}$ we need to use.

We can exploit our knowledge that the total space Y_{n+1} has to be Calabi-Yau in the following way: It is known from the general theory of elliptic fibrations that the chern classes of B_n and Y_{n+1} are related in the following way:

$$c_1\left(Y_{n+1}\right) \simeq \pi^* \left(c_1\left(B_n\right) - \sum_i \frac{\delta_i}{12} [\Gamma_i] \right) \tag{4.22}$$

where $\pi : Y_{n+1} \to B_n$ is the fibration and the discriminant Δ vanishes along the divisors Γ_i to order δ_i. $[\Gamma_i]$ is the Poincaré dual of the divisor Γ_i as usual. This equation holds up to terms involving degenerations of higher codimension which are irrelevant for the following [5].

Since $c_1(Y_{n+1})$ needs to be zero, we immediately get

$$12 c_1\left(B_n\right) = \sum_i \delta_i \left[\Gamma_i\right] = [\Delta] \; ; \tag{4.23}$$

by customary abuse of notation we have written $[\Delta]$ for the Poincaré dual of the zero locus $Z_{\mathcal{L}^{12}}(\Delta)$ of the section $\Delta \in H^0(B_n, \mathcal{L}^{12})$. This result reminds us of the charge cancellation condition in perturbative type IIB theory, $\sum_i N_i \left[\Gamma_i\right] = 4 \left[O_7\right]$: That condition is now automatically incorporated in the F-theory geometry!

The condition (4.23) is solved by taking $\mathcal{L}$ to be the anticanonical bundle $K^*_{B_n}$. The reason is that $\mathcal{O}(\Delta) = \mathcal{L}^{12}$ and thus, using (2.34),

$$[\Delta] = c_1(\mathcal{O}(\Delta)) = c_1(\mathcal{L}^{12}) = 12\, c_1(\mathcal{L}) \; .$$

Plugging in $\mathcal{L} = K^*_{B_n}$ proves the claim $[\Delta] = -12\, c_1(K_{B_n}) = 12\, c_1(B_n)$. Let us summarize:

$$f \in H^0(B_n, K^{-4}_{B_n}) \qquad \text{and} \qquad g \in H^0(B_n, K^{-6}_{B_n}) \tag{4.24}$$

and because of homogeneity of (4.19) also the coordinates transform as sections of the base,

$$x \in H^0(B_n, K^{-2}_{B_n}) \; , \qquad y \in H^0(B_n, K^{-3}_{B_n}) \qquad \text{and} \qquad z \in H^0(B_n, \mathcal{O}) \; . \tag{4.25}$$

(Of course, they additionally transform as sections of the respective line bundles $\mathcal{O}(w_i)$ over $\mathbb{P}^2_{(2:3:1)}$.)

D7-Branes. After the discussion above, we expect a zero locus Γ_i where the discriminant vanishes to order δ_i to describe a stack of δ_i D7-branes. Let us do one more sanity check and see how the complex structure of the torus looks close to Γ_i. Let w be the coordinates normal to Γ_i, then $\Delta \sim w^{\delta_i}$ close to the zero locus. Using (4.21) we see that

$$j(\tau) \sim w^{-\delta_i} \; .$$

We haven't given a rigorous definition of the j-invariant, for us it will suffice to know that it has the expansion

$$j(z) = \mathrm{e}^{-2\pi \mathrm{i}\, z} + 744 + 196\,884\, \mathrm{e}^{2\pi \mathrm{i}\, z} + \cdots \; . \tag{4.26}$$

Taking only the first term of the expansion this results in

$$\tau \sim \frac{\delta_i}{2\pi \mathrm{i}} \, \ln w \; , \tag{4.27}$$

which is what we expected. This works not only for D7-branes but in general also for $[p,q]$-branes, see [5] on how their location is encoded in the vanishing loci of (p,q)-cycles of the torus. The fact that every $[p,q]$-brane locally looks like a D7-brane corresponds to the fact that we can always locally choose a basis of $H^2(T^2, \mathbb{Z})$ so that the degenerate cycle corresponds to the $(1,0)$-fiber.

ord(f)	ord(g)	ord(Δ)	Fiber Type	Surface Singularity	Monodromy
≥ 0	≥ 0	0	smooth	none	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
0	0	n	I_n	A_{n-1}	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
2 ≥ 2	≥ 3 3	$n+6$ $n+6$	I_n^*	D_{n+4}	$-\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
≥ 1	1	2	II	none	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
≥ 4	5	10	II^*	E_8	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$
1	≥ 2	3	III	A_1	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
3	≥ 5	9	III^*	E_7	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
≥ 2	2	4	IV	A_2	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$
≥ 3	4	8	IV^*	E_6	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

Table 4.1: The classification of singular fibers of an elliptic surface in dependency of the vanishing degrees of f, g and Δ. Also given is the corresponding monodromy and the type of surface singularity. Taken in this form from [10, Table 2.2].

Example: $K3$. As an example we consider an elliptic $K3$ surface, i.e. a Calabi-Yau total space Y_2 together with an elliptic fibration $\pi : Y_2 \to B_1 = \mathbb{P}^1$. Let $\{u_0, u_1\}$ be homogeneous coordinates of $\mathbb{P}^1$, the total space Y_2 can then be described by the Weierstrass equation in an ambient space with the coordinates

$$\begin{array}{ccccc} u_0 & u_1 & x & y & z \\ \hline 0 & 0 & 2 & 3 & 1 \\ 1 & 1 & 4 & 6 & 0 \end{array} . \tag{4.28}$$

We learned above that because of the Calabi-Yau requirement the discriminant transforms as a section of $K_{\mathbb{P}^1}^{-12} = \mathcal{O}(-2)^{-12} = \mathcal{O}(24)$ with respect to the base. We know from subsection 2.2.1 that it therefore is a polynomial of degree 24 in u_0 and u_1 having 24 zeroes.

4.4 Singularities and Gauge Groups

As we have seen, the locus of the fiber degeneration corresponds to the position of stacks of D7-branes and the vanishing order of the discriminant gives the amount of branes. We already know that a stack of N D7-branes leads to a U(N) gauge theory in the low energy limit via Chan-Paton factors. The appearance of $[p, q]$-branes adds something new to this picture: The fact that a $\binom{p}{q}$-string can only end on a $[p, q]$-brane can obviously change the gauge group.

We hence have to study singularities of the elliptic fibration. [5, Ch. 2.5-3.3] and [10, Ch. 2.3] provide excellent write-ups, we are only going to summarize the most important points here.

Gauge Symmetry from Degenerations. The first step is to classify the possible singularities that can occur. This was done by Kodaira in 1963 for the case of elliptic surfaces (fibrations $Y_2 \to B_1$) [34]. The type of the fiber singularity depends only on the vanishing degrees of f, g and Δ and one can also directly read off the surface singularity type and the induced monodromy, see table 4.1.

The general idea about what the connection between a singularity of the surface over the divisor $S \subset B_n$ and the gauge group G is, is as follows: We need to resolve the singularity, for example with a *split-simultaneous resolution* $\bar{Y}_{n+1}$. In this process, the singular fiber is replaced by a tree of $\mathbb{P}^1$s which we will call $\mathbb{P}^1_i$ ($i \in \{1, \ldots, \mathrm{rk}\, G\}$). Those $\mathbb{P}^1_i$ intersect one another like the simple roots of the Lie algebra corresponding to G:

Let $\mathbb{P}^1_i \to E_i \to S$ be $\mathbb{P}^1_i$ fibered over the divisor S, E_i is a divisor of $\bar{Y}_{n+1}$ and let $E_0 = \hat{S} - \sum_i a_i E_i$ (where $\hat{S}$ is the elliptic fibration over S). Then

$$\int_{\bar{Y}_{n+1}} [E_i] \wedge [E_j] \wedge \pi^* \tilde{\omega} = \tilde{C}_{ij} \int_S \tilde{\omega} \tag{4.29}$$

for any $\tilde{\omega} \in H^{2n-2}(B_n)$, where $\tilde{C}$ is the (extended) Cartan matrix of G, see section 2.3.

The actual manifold Y_n has to be seen as the limit of zero volume of the $\mathbb{P}^1_i$. We can understand the appearance of the gauge bosons of the group G from the F/M-theory duality: The M-theory 3-form C_3 and the M2-brane are reduced along the $\mathbb{P}^1_i$, giving exactly $\dim G$ states corresponding to the algebra generators.

The non-Abelian part of the gauge group in F-theory compactifications is identical to the singularities of the elliptic fibration as indicated in table 4.2 below [10].

Tate Models. Note that the classification in table 4.1 which is only valid for elliptic surfaces contains only singularity types A, D and E corresponding to *simply laced Dynkin diagrams* (see figure 2.2). (It is called *ADE classification* for that reason.) Once we go to elliptic fibrations of higher dimension, this situation changes: Monodromies along the brane can "fold" the Dynkin diagram such that non-simply laced Lie algebras emerge.

There is an algorithm [35], called Tate's algorithm, which allows to read off the type of fiber in the general case. It was actually derived for elliptic CY 3-folds only, but no complete classification of the more general situation exists as of yet [5]. In Tate's formalism, we bring equation (4.19) into *Tate form*

$$P_W = x^3 - y^2 + a_1\, xyz + a_2\, x^2 z^2 + a_3\, yz^3 + a_4\, xz^4 + a_6\, z^6 = 0\ . \tag{4.30}$$

This is always possible locally, but not necessarily globally, in other words the a_i might be only local sections of $K_{B_n}^{-i}$. We will see that global Tate models are especially convenient. Obviously, every global Tate model defines a Weierstrass model, but the converse is not true.

For later use, let us quickly state the relationship between the a_i and f and g:

$$\begin{aligned} &f = -\frac{1}{48}\left(\beta_2^2 - 24\,\beta_4\right), \quad g = -\frac{1}{864}\left(-\beta_2^3 + 36\,\beta_2\beta_4 - 216\,\beta_6\right)\ , \\ &\beta_2 = a_1^2 + 4\,a_2, \quad \beta_4 = a_1 a_3 + 2\,a_4, \quad \beta_6 = a_3^2 + 4\,a_6\ . \end{aligned} \tag{4.31}$$

From the vanishing degrees of the a_i and of Δ we can read off the singularity type and the gauge group, see table 4.2. In the table, we can see the important distinction between "non-split" configurations (ns), where monodromies occur and fold the Dynkin diagrams, and "split" configurations (s) where this does not happen. Another special case is the "semi-split" configuration $I_{2k}^{*\mathrm{ss}}$, for details see [35]. Using Tate's algorithm, we can conveniently just look at the vanishing degrees of the different sections and read off which configuration we are in.

Example: SU(5) **Gauge Group.** For example, if we want a SU(5) gauge group along the divisor $S: w = 0$, the vanishing orders in w can be read off in the I_{2k+1}^{s} line of the table for $k = 2$. This corresponds to

$$a_1 = \mathfrak{b}_5, \quad a_2 = \mathfrak{b}_4\, w, \quad a_3 = \mathfrak{b}_3\, w^2, \quad a_4 = \mathfrak{b}_2\, w^3, \quad a_6 = \mathfrak{b}_0\, w^5\ , \tag{4.32}$$

where the sections $\mathfrak{b}_i$ do not contain global factors of w. Using equations (4.20) and (4.31), we find [5]

$$\Delta = -w^5 \underbrace{\left(\mathfrak{b}_5^4 P + w\,\mathfrak{b}_5^2(8\mathfrak{b}_4 P + \mathfrak{b}_5 R) + \mathcal{O}(w^2)\right)}_{S_1} \tag{4.33}$$

for some polynomials $P(\mathfrak{b}_0 \dots \mathfrak{b}_5)$ and $R(\mathfrak{b}_0 \dots \mathfrak{b}_5)$. The expression S_1 describes the locus of an I_1 singularity, in cohomology $[\Delta] = 5[S] + [S_1]$.

Singularity type	Gauge group	Dynkin diagram	Vanishing degrees a_1	a_2	a_3	a_4	a_6	Δ
I_0	—		0	0	0	0	0	0
I_1	—		0	0	1	1	1	1
I_2	SU(2)	A_1	0	0	1	1	2	2
I_{2k}^{ns}	SP($2k$)	C_{2k}	0	0	k	k	$2k$	$2k$
I_{2k}^{s}	SU($2k$)	A_{2k-1}	0	1	k	k	$2k$	$2k$
I_{2k+1}^{ns}	[unconv.]		0	0	$k+1$	$k+1$	$2k+1$	$2k+1$
I_{2k+1}^{s}	SU($2k+1$)	A_{2k}	0	1	k	$k+1$	$2k+1$	$2k+1$
II	—		1	1	1	1	1	2
III	SU(2)	A_1	1	1	1	1	2	3
IV^{ns}	[unconv.]		1	1	1	2	2	4
IV^{s}	SU(3)	A_2	1	1	1	2	3	4
$I_0^{*\mathrm{ns}}$	G_2	G_2	1	1	2	2	3	6
$I_0^{*\mathrm{ss}}$	SO(7)	B_3	1	1	2	2	4	6
$I_0^{*\mathrm{s}}$	SO(8)	D_4	1	1	2	2	4	6
$I_1^{*\mathrm{ns}}$	SO(9)	B_4	1	1	2	3	4	7
$I_1^{*\mathrm{s}}$	SO(10)	D_5	1	1	2	3	5	7
$I_2^{*\mathrm{ns}}$	SO(11)	B_5	1	1	3	3	5	8
$I_2^{*\mathrm{s}}$	SO(12)	D_6	1	1	3	3	5	8
$I_{2k-3}^{*\mathrm{ns}}$	SO($4k+1$)	B_{2k}	1	1	k	$k+1$	$2k$	$2k+3$
$I_{2k-3}^{*\mathrm{s}}$	SO($4k+2$)	D_{2k+1}	1	1	k	$k+1$	$2k+1$	$2k+3$
$I_{2k-2}^{*\mathrm{ns}}$	SO($4k+3$)	B_{2k+1}	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
$I_{2k-2}^{*\mathrm{s}}$	SO($4k+4$)	D_{2k+2}	1	1	$k+1$	$k+1$	$2k+1$	$2k+4$
$IV^{*\mathrm{ns}}$	F_4	F_4	1	2	2	3	4	8
$IV^{*\mathrm{s}}$	E_6	E_6	1	2	2	3	5	8
III^*	E_7	E_7	1	2	3	3	5	9
II^*	E_8	E_8	1	2	3	4	5	10
non-min	—		1	2	3	4	6	12

Table 4.2: Refined Kodaira classification resulting from Tate's algorithm [35].

Matter Curves. There are two ways how chiral charged matter can arise in F-theory: The first are so-called bulk states that propagate along the whole divisor. Bulk states will be irrelevant for what follows, for a discussion we refer to [5].

The second possibility comes up at the loci where two singular divisors D_a and D_b intersect. If we do F-theory on an elliptic CY 4-fold fibered over B_3, a divisor is a two-dimensional subvariety of B_3 and the intersection $C_{ab} = D_a \cap D_b$ of two divisors is a curve, such curves are called *matter curves*.

Matter comes into play in the following way: Along the matter curve, the two sets of $\mathbb{P}^1$s intersect and the gauge symmetry is enhanced to a new gauge group G_{ab} with

$$\operatorname{rk} G_{ab} = \operatorname{rk} G_a + \operatorname{rk} G_b \; . \tag{4.34}$$

If we look at how the M-theory C_3 and the M2-brane can be reduced along the $\mathbb{P}^1$s we see that there is matter transforming in the adjoint of G_{ab}. Considering how this decomposes under the subduction $G_{ab} \to G_a \times G_b$, we generically get

$$\mathbf{ad}_{G_{ab}} \to (\mathbf{ad}_{G_a}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{ad}_{G_b}) \oplus \bigoplus_x (\mathbf{R}_x, \mathbf{U}_x) \tag{4.35}$$

for some representations $\mathbf{R}_x$ and $\mathbf{U}_x$. We see that there are extra states appearing at the intersection of the divisors.

Matter in our SU(5) **Example.** Let's go back to our example (4.33). At the intersection locus of the I_5^{s} singularity S (with gauge group A_4 of rank 4) and the I_1 singularity S_1, the singularity can be

	Sing. type	Vanishing degrees a_1	a_2	a_3	a_4	a_6	Δ	Gauge enh. Type	Group	Object equation
GUT:	I_5^{s}	0	1	2	3	5	5	A_4	SU(5)	$S : w = 0$
Matter:	I_6^{s}	0	1	3	3	6	6	A_5	SU(6)	$P_5 : P = 0$
	$I_1^{*\mathrm{s}}$	1	1	2	3	5	7	D_5	SO(10)	$P_{10} : \mathfrak{b}_5 = 0$
Yukawa:	$I_2^{*\mathrm{s}}$	1	1	3	3	5	8	D_6	SO(12)	$\mathfrak{b}_3 = 0 = \mathfrak{b}_5$
	$IV^{*\mathrm{s}}$	1	2	2	3	5	8	E_6	E_6	$\mathfrak{b}_4 = 0 = \mathfrak{b}_5$
Extra:	I_7^{s}	0	1	3	4	7	7	A_6	SU(7)	$P = 0 = R$, $(\mathfrak{b}_4, \mathfrak{b}_5) \neq (0,0)$

Table 4.3: Relevant gauge enhancements in an SU(5) GUT geometry. Taken in this form from [5, Table 2].

enhanced either to I_6^{s} with gauge group A_5 or to $I_1^{*\mathrm{s}}$ with gauge group D_5 [36, Ch. 2.2].

- The enhancement $A_4 \to A_5$ corresponds to SU(5) $\to$ SU(6) with branching rule

$$\mathbf{35} \to \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{5}_1 + \bar{\mathbf{5}}_{-1} \; . \tag{4.36}$$

  The matter curve hosts matter in the $\mathbf{5} = \square$. According to table 4.2, it occurs when the vanishing degree of Δ is 6 which happens when $P = 0$.

$$P_5 : w = 0 \quad \cap \quad P = 0 \; . \tag{4.37}$$

- The enhancement $A_4 \to D_5$ corresponds to SU(5) $\to$ SO(10) with branching rule

$$\mathbf{45} \to \mathbf{24}_0 + \mathbf{1}_0 + \mathbf{10}_2 + \overline{\mathbf{10}}_{-2} \; . \tag{4.38}$$

  The matter curve hosts matter in the $\mathbf{10} = \square\square$, it occurs when $\mathfrak{b}_5 = 0$ such that the discriminant scales as w^7:

$$P_{10} : w = 0 \quad \cap \quad \mathfrak{b}_5 = 0 \; . \tag{4.39}$$

Yukawa Points. At a point where two matter curves intersect, there is a further gauge enhancement. This realizes a Yukawa interaction between the matter states living on the matter curves.

In our SU(5) example there are three possibilities:

- An E_6 enhancement $\mathfrak{b}_4 = 0 = \mathfrak{b}_5$ gives a $\mathbf{10}\,\mathbf{10}\,\mathbf{5}$ coupling.
- An F_6 enhancement $\mathfrak{b}_3 = 0 = \mathfrak{b}_5$ gives a $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$ coupling.
- An A_6 enhancement happens where $P = 0 = R$ but $(\mathfrak{b}_4, \mathfrak{b}_5) \neq (0,0)$. This gives a $\mathbf{5}\,\bar{\mathbf{5}}\,\mathbf{1}$ coupling with an extra GUT singlet.

The various gauge enhancement we have discussed in our SU(5) model are summarized in table 4.3.

4.5 The Sen Limit

Sen's Weak Coupling Limit (following [10]). To connect F-theory to type IIB orientifolds, one needs to find configurations where the imaginary part of τ is large almost everywhere. (This automatically gives us an SL$(2,\mathbb{Z})$-frame where all monodromies are the ones of D-branes or O-planes, see the discussion in [10, Ch. 2.4].) A procedure giving us exactly such a limit was presented by Sen in [37].

We start from the Weierstrass form (4.19), putting the polynomial P_W into a similar form like in the first line of (4.31). It is however customary[2] to rename the sections, let $b_2 = -\frac{1}{4}\,\beta_2$, $b_4 = \frac{1}{4}\,\beta_4$ and

[2] We'll follow the conventions of [1]. Other authors also use $h = -\frac{1}{3}\,b_2$, $\eta = 2\,b_4$ and $\chi = -12\,b_6$.

$b_6 = -\frac{1}{48}\,\beta_6$, then $f = -\frac{1}{3}\,b_2^2 + 2\,b_4$ and $g = \frac{2}{27}\,b_2^3 - \frac{2}{3}\,b_2 b_4 + b_6$. Now we rescale those sections with a parameter t,

$$b_2 \to t^0\, b_2, \quad b_4 \to t^1\, b_4, \quad b_6 \to t^2\, b_6\ , \tag{4.40}$$

such that

$$f = -\frac{1}{3}\,b_2^2 + 2t\,b_4, \quad g = \frac{2}{27}\,b_2^3 - \frac{2}{3} t\,b_2 b_4 + t^2\,b_6 \quad \text{and} \quad \Delta = -4t^2\,b_2^2\left(b_4^2 - b_2 b_6\right) + \mathcal{O}(t^3)\ . \tag{4.41}$$

Remembering from (4.21) that $j(\tau) = \frac{4(24\,f)^3}{\Delta}$ we see that in the limit $t \to 0$, $|j|$ goes to infinity and therefore the coupling becomes weak as desired. The fibration degenerates at the loci

$$b_2 = 0 \qquad \text{and} \tag{4.42}$$

$$b_4^2 = b_2 b_6\ . \tag{4.43}$$

Closer inspection shows that the monodromy around $b_2 = 0$ is $-T^{-4}$ so that this is an O-plane. Further, we can choose a frame where the monodromies around the $b_4^2 = b_2 b_6$ branes are all equal to T, so that (4.43) describes the positions of D-branes.

We can finally construct the type IIB Calabi-Yau by building a double cover of the base B_3 which is branched over the locus of the O-plane. Concretely, we consider the equation

$$\xi^2 = b_2 \tag{4.44}$$

where ξ is a section of $K^*_{B_n}$. The complete intersection described by the equation (4.44) in this line bundle is a Calabi-Yau 3-fold $\hat{B}_3$, it is the one the type IIB theory is defined on. The D-branes are located at

$$b_4^2 = \xi^2\, b_6\ . \tag{4.45}$$

The Stable Version. A refined version of this procedure was recently published [38], we'll follow [1,39]. Let us plug (4.41) into the Weierstrass equation (4.19) again, with the slight change that we replace x by the coordinate $s = x + h\,z^2$. That gives the equation

$$W_5: \quad y^2 = s^3 + b_2\,s^2 z^2 + 2b_4\,tsz^4 + b_6\,t^2 z^6\ . \tag{4.46}$$

This is a family of CY 4-folds parametrized by t, which can also be seen as a 5-fold W_5 given by equation (4.46) in the ambient 6-fold. The 6-fold is described by the three coordinates u_i of the base and the four coordinates s, y, z, t together with the scaling relation

s	y	z	t
2	3	1	0
K_B^{-2}	K_B^{-3}	$\mathcal{O}$	$\mathcal{O}$

and the SR ideal $\langle syz \rangle$. (In the last line we also list how the coordinates transform as sections of the base B.)

At $t = 0$, the fiber of the 4-fold degenerates everywhere. This is not a stable singularity since also W_5 becomes singular at $s = y = t = 0$. If we do not want to lose information in the limit $t \to 0$, we need to blow up this singular locus. This can be done exactly as described in subsection 2.2.3: We introduce one additional coordinate λ to describe the blown up 6-fold, with scaling relations

s	y	z	t	λ
2	3	1	0	0
1	1	0	1	-1
K_B^{-2}	K_B^{-3}	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$

and SR ideal $\langle syz, syt, z\lambda \rangle$. The blow-up map is

$$\sigma(s : y : z : t : \lambda) = (s\lambda : y\lambda : z : t\lambda) \tag{4.47}$$

showing that equation (4.46) now reads (after dividing by λ^2)

$$\tilde{W}_5: \quad y^2 = \lambda s^3 + b_2\,s^2 z^2 + 2b_4\,tsz^4 + b_6\,t^2 z^6\ . \tag{4.48}$$

IIB Data from $\tilde{W}_5$. We are interested in the central fiber, $\sigma^{-1}(t=0)$. This is a 4-fold consisting of two components, because either t or λ can be set to zero in the resolved $\tilde{W}_5$. Those two components are

$$W_T: \quad \tilde{W}_5 \cap \{t=0\}: \quad y^2 = s^2\left(\lambda s + b_2\, z^2\right) \ , \tag{4.49}$$
$$W_E: \quad \tilde{W}_5 \cap \{\lambda=0\}: \quad y^2 = b_2\, s^2 z^2 + 2b_4\, tsz^4 + b_6\, t^2 z^6 \ . \tag{4.50}$$

The intersection $W_T \cap W_E$ is a 3-fold where $t = 0 = \lambda$ and therefore given by the equation

$$X_3: \quad \tilde{W}_5 \cap \{t=0=\lambda\}: \quad y^2 = b_2\, s^2 z^2 \ . \tag{4.51}$$

Due to the SR ideal, the divisors $\{z=0\}$ and $\{s=0\}$ do not meet X_3 and we can define $\xi = y/(zs)$. Now the equation reads $\xi^2 = b_2$, we already know this type of equation from (4.44): It describes the CY 3-fold on which the type IIB theory is defined!

We continue and analyze W_E a bit more. Since $\lambda = 0$, we can set $z = 1$ and then W_E is described by

$$W_E: \quad y^2 = b_2\, s^2 + 2b_4\, ts + b_6\, t^2 \ . \tag{4.52}$$

This is an equation in an ambient 5-fold described by the coordinates of the base and the coordinates s, y, t of equal weights

s	y	t
1	1	1
K_B^{-2}	K_B^{-3}	$\mathcal{O}$

with SR ideal $\langle syt \rangle$. In other words, W_E is a fibration over B_3 and each fiber of it is described by a quadratic equation in a $\mathbb{P}^2$. This is called a conic bundle. Generically, those fibers are $\mathbb{P}^1$s, except at the singularity where the discriminant

$$\Delta_E = b_2 b_6 - b_4^2 \tag{4.53}$$

vanishes and the fiber consists of two $\mathbb{P}^1$s. We recognize this to be the locus of the D7-brane like in (4.45).

The Cylinder. For future reference, we define the *cylinder*

$$R_3 = W_E \cap \{\Delta_E = 0\} \ , \tag{4.54}$$

it is a 3-fold which consists of the two $\mathbb{P}^1$s fibered over the D7-brane.

Chapter 5

Hypercharge Flux

5.1 Interlude: Stückelberg Mass Terms

Massive $\mathrm{U}(1)$ **Gauge Bosons.** Let us quickly recap the theory of a free massless spin-1 boson A_μ [17]: It is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.1}$$

which is invariant under the gauge symmetry

$$A_\mu \to A_\mu + \frac{1}{g}\,\partial_\mu \alpha \ . \tag{5.2}$$

This is a special case of (A.84), but here α simply takes values in $\mathfrak{u}(1) = \mathbb{R}$.

We can eliminate the gauge symmetry by adding a gauge fixing term $-\frac{\lambda}{2}(\partial A)^2$ such that the field is in Lorenz gauge $\partial A = 0$. For $\lambda = 1$ (Feynman gauge), the gauge-fixed Lagrangian reads

$$\mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu \ . \tag{5.3}$$

When we quantize the theory, we still have to impose the constraint $\langle \partial A \rangle = 0$ after quantization (*Gupta-Bleuler condition*). This gives us a well-defined quantum theory of two degrees of freedom (one ghost mode is removed by the Gupta-Bleuler condition, another spurious mode decouples from all physical processes).

A mass term $\frac{m^2}{2} A^\mu A_\mu$ destroys gauge symmetry, adding it yields the *Proca Lagrangian*

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A^\mu A_\mu \ . \tag{5.4}$$

The absence of gauge symmetry does not hurt since the constraint $\partial A = 0$ follows from the equation of motion $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$. After quantization and imposing that constraint, we arrive at a theory of three positive-norm degrees of freedom.

There are however mechanisms than can realize a massive gauge boson with $\mathrm{U}(1)$ gauge symmetry if we add interactions with other degrees of freedom (see [40] for a review).

The Higgs Mechanism. The most well-known one (realized in the Standard Model, see subsection 3.1.1) is the *Higgs mechanism*: We add a complex scalar field Φ with a potential V exhibiting spontaneous symmetry breaking $\langle \Phi \rangle = v \neq 0$. Φ transforms in the fundamental representation

$$\Phi \to \mathrm{e}^{-\mathrm{i}\alpha}\, \Phi \tag{5.5}$$

of $\mathrm{U}(1)$ with the covariant derivative $D_\mu \Phi = (\partial_\mu + \mathrm{i} g\, A_\mu)\, \Phi$. Writing down the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^* (D^\mu \Phi) - V(|\Phi|^2) \tag{5.6}$$

and expanding $\Phi = v + \varphi$, we immediately see that there is a mass term $\mathcal{L} \supset -g^2 v^2 \Phi^2$.

To make this more explicit, we expand Φ in terms of amplitude and phase,

$$\Phi = \frac{1}{\sqrt{2}}(v + f)\, \mathrm{e}^{-\mathrm{i}\Theta/v} , \tag{5.7}$$

and we get [17]

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2}(\partial f)^2 + \frac{g^2 v^2}{2}\left(A - \frac{\partial\Theta}{gv}\right)^2 - \frac{gv^2}{2} f^2 + \mathcal{O}(f^3) . \tag{5.8}$$

Choosing the gauge $\Theta \equiv 0$ this is the theory of a massive real scalar f and a massive vector field A. One says that A has "eaten" one of the degrees of freedom of Φ so that it could become a massive field with three real degrees of freedom.

The Classical Stückelberg Mechanism. Another perspective is this one: Consider again (5.8) and take $g \to 0$ and $v \to \infty$ such that $gv = m$ stays constant. The mass $m_f^2 = gv^2$ of the scalar field f goes to infinity, we can leave it out and are left with

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}\left(A_\mu - \frac{1}{m}\partial_\mu\Theta\right)^2 . \tag{5.9}$$

This is the *Stückelberg Lagrangian*. Since we are still free to choose a gauge, $\Theta \equiv 0$ say, this describes only the three degrees of freedom of the massive vector boson.

We could have written down this Lagrangian just as well without any prior reference to the Higgs mechanism: Just take an *axion* Θ with shift symmetry $\Theta \to \Theta + \frac{m}{g}\alpha$, then $A_\mu - \frac{1}{m}\partial_\mu\Theta$ is invariant under the combined gauge symmetry. The transformation behavior of Θ is called a nonlinear (affine) representation of U(1).

Note that in the case of non-Abelian gauge symmetries, gauge symmetry is necessary for Faddeev-Popov quantization. Further, the Stückelberg mechanism does not give a well-defined quantum theory: The resulting theory is either not unitary or not renormalizable [41]. The only possibility that is left is the Higgs mechanism.

The Stückelberg Mechanism in String Theory. We are interested in one particular instance in which Stückelberg masses play a role: Consider again a vector boson with $F = \mathrm{d}A$, and additionally a 2-form field B with field strength $h = \mathrm{d}B$. A coupling $\sim \int B \wedge F$ in $\mathbb{R}^{1,3}$ will give the vector boson a Stückelberg mass [42, Ch. 9.5.2].

Toward proving this fact, we will write down the relevant parts of the action. It consists of the kinetic terms for A and B and of the interaction $\int B \wedge F$:

$$S[A, B] = -\frac{1}{2}\int_{\mathbb{R}^{1,3}} \left(h \wedge *h + \frac{1}{g^2}F \wedge *F + 2c\, B \wedge F\right) , \tag{5.10}$$

in components (using (2.14))

$$S[A, B] = \int_{\mathbb{R}^{1,3}} \left(-\frac{1}{12}h^{\mu\nu\rho}h_{\mu\nu\rho} - \frac{1}{4g^2}F^{\mu\nu}F_{\mu\nu} + \frac{c}{4}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}F_{\rho\sigma}\right) \mathrm{vol} . \tag{5.11}$$

A common trick when working with a 2-form field in four dimensions is to write the action completely in terms of its field strength while enforcing $\mathrm{d}h = 0$ with a Lagrange multiplier η. More explicitly, we perform a partial integration $B \wedge F = B \wedge \mathrm{d}A = \mathrm{d}(B \wedge A) - h \wedge A$ and arrive at

$$S[A, B] = -\frac{1}{2}\int \left(h \wedge *h + \frac{1}{g^2}F \wedge *F - 2c\, h \wedge A\right)$$

assuming that the boundary terms vanish. Introducing the Lagrange multiplier we get the equivalent action

$$S[A, h, \eta] = -\frac{1}{2}\int \left(h \wedge *h + \frac{1}{g^2}F \wedge *F\right) + c\int h \wedge A - \int \eta \wedge \mathrm{d}h , \tag{5.12}$$

in components

$$S[A,h,\eta] = \int \left(-\frac{1}{12} h^{\mu\nu\rho} h_{\mu\nu\rho} - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - \frac{c}{6} \epsilon^{\mu\nu\rho\sigma} h_{\mu\nu\rho} A_\sigma + \frac{1}{6} \eta\, \epsilon^{\mu\nu\rho\sigma} \partial_\mu h_{\nu\rho\sigma} \right) \mathrm{vol} \ . \tag{5.13}$$

We perform another partial integration $\eta \wedge \mathrm{d}h = \mathrm{d}(\eta \wedge h) + h \wedge \mathrm{d}\eta$ eliminating the only appearing derivative of h. In the resulting action,

$$S[A,h,\eta] = -\frac{1}{2} \int \left(h \wedge *h + \frac{1}{g^2} F \wedge *F \right) + \int h \wedge (c\, A - \mathrm{d}\eta) \ , \tag{5.14}$$

h acts as a Lagrange multiplier enforcing the constraint $\frac{\delta S}{\delta h} = 0$. As

$$\delta_h S = \int \delta h \wedge (-*h + c\, A - \mathrm{d}\eta) \ ,$$

that constraint reads $h = *(c\, A - \mathrm{d}\eta)$ or $h^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho\sigma} \left(c\, A_\sigma - \partial_\sigma \eta \right)$.

Plugging this result back into (5.12) finally yields the classical Stückelberg action:

$$S[A,\eta] = \int \left[-\frac{1}{2g^2} F \wedge *F + \frac{1}{2} \left(c\, A - \mathrm{d}\eta \right) \wedge * \left(c\, A - \mathrm{d}\eta \right) \right] \ , \tag{5.15}$$

in components

$$S[A,\eta] = \int \left[-\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \left(c\, A_\mu - \partial_\mu \eta \right)^2 \right] \mathrm{vol} \ . \tag{5.16}$$

We already know that this gives a mass to the vector boson A, this concludes our proof.

5.2 SU(5) GUT Breaking via Hypercharge Flux

We have seen in section 4.4 how we can build an SU(5) GUT model using F-theory. In order to build a realistic model, this GUT group still has to be broken down to the Standard Model gauge group. One idea for how to achieve this would be to include a GUT Higgs in the **24** of SU(5) as discussed in subsection 3.2.3. We end up with a traditional four-dimensional GUT model which has the same problems (like proton decay or doublet-triplet splitting) that are outlined at the same place, though.

It would be better to look for an intrinsically stringy mechanism so that no low-energy SU(5) GUT theory ever arises and those problems can be circumvented. In fact, there are several possible approaches to do so. In this text, we will pursue the following appealing option: We already mentioned in subsection 3.3.1 that we can include background fluxes in our compactification, that is vacuum expectation values of certain field strengths. In the language of type IIB string theory, it is intuitively clear that turning on a background flux

$$\mathfrak{su}(5) \supset \langle A_\mu \rangle = c_\mu, \quad c^2 \neq 0 \tag{5.17}$$

will reduce the gauge symmetry to those transformations that leave the background flux unchanged: The symmetry is broken to the commutant of $\langle A_\mu \rangle$ in $\mathfrak{su}(5)$. In the following we will prove this fact explicitly and then use it to build a realistic model: The commutant of the hypercharge generator in $\mathfrak{su}(5)$ is exactly the Standard Model, this is obvious from the definitions in subsection 3.2.3. When we turn on a hypercharge flux, we will have to take great care that the hypercharge generator remains massless, though.

For a more detailed discussion on how GUTs can be broken in string theory see [5, Ch. 4.2] or [43, Ch. 2]. The phenomenon of GUT breaking via hypercharge flux was first discovered in [2]. One can find a good treatment of this mechanism in [3,8], see also the introduction in [1]. For example [43,44] describe how to implement this directly in F-theory.

Symmetry Breaking via Background Fluxes. We want to prove the claim that a Yang-Mills background flux (5.17) breaks the symmetry to the commutant of the flux, but need to introduce some standard notation first. Let $\{T^a\}$ be the generators of SU(5) like in subsection 2.3.1 such that $[T^a, T^b] = f^{abc}T^c$ and $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. Then we can expand $A_\mu = A^a_\mu T^a$ as well as $F_{\mu\nu} = F^a_{\mu\nu}T^a$ in this basis and (A.86) becomes

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \mathrm{i}gf^{abc}\,A^b_\mu A^c_\nu \ . \tag{5.18}$$

We now go back to the DBI action given in (3.36) for a single brane. For a stack of D-branes we need to add a trace over the internal indices, to lowest order in F it reads[1] (see e.g. [22, Ch. 16.5])

$$S_{\mathrm{DBI}} = -\kappa\int_{D_7} \mathrm{tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\ \mathrm{d}^8\xi + \cdots = -\kappa\int_{D_7}\frac{1}{8}F^a_{\mu\nu}F^{\mu\nu\,a}\ \mathrm{d}^8\xi + \cdots \tag{5.19}$$

(with $\kappa = \frac{(2\pi)^3}{\ell_s^4}$ a constant). For definiteness we will directly assume a flux in the hypercharge direction,

$$\langle A_\mu\rangle = c_\mu = c^Y_\mu\,T^Y \tag{5.20}$$

(where T^Y is the matrix that was called Y_1 in subsection 3.2.3).

The background value leads to mass terms, the only relevant term in the Lagrangian is the one with no derivatives,

$$\kappa\frac{g^2}{8}f^{abd}\,A^b_\mu A^d_\nu\,f^{ade}\,A^{\mu\,c}A^{\nu\,e} \ .$$

The mass matrix is

$$M^{bc} = \left.\frac{\partial^2\mathcal{L}}{\partial A^b_\mu\partial A^{\mu\,c}}\right|_{A=\langle A\rangle} = \frac{\kappa}{2}g^2c^2\,f^{abY}f^{acY} = g^2c^2\ \mathrm{tr}\left([T^b, T^Y]\cdot[T^c, T^Y]\right)\ . \tag{5.21}$$

Note that in particular terms of the form $\sim(\partial_\nu A^a_\rho)f^{ade}A^{\nu\,d}A^{\rho\,e}$ do not lead to masses because $\frac{\partial^2}{\partial A^b_\mu\partial A^{\mu\,c}}$ gives zero (the structure constants f^{abc} are antisymmetric).

The gauge bosons remaining massless are exactly the ones in the kernel of the mass matrix, which is precisely the commutant of T^Y in $\mathfrak{su}(5)$.

Approaching the Standard Model Gauge Group. We already discussed above that the commutant of T^Y in $\mathfrak{su}(5)$ is the Standard Model $\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)_Y$. We have thus learned: By turning on a hypercharge flux, the SU(5) GUT is broken down at least to the Standard Model.

But we still need to make sure that the generators of the Standard Model gauge group do not acquire a mass from the Chern-Simons coupling (3.37). In the following we will see that the method we have used generically makes the hypercharge generator massive, but we can get around that by choosing the value of the hypercharge flux in a clever way.

Stückelberg Mass Terms for the Hypercharge Generator. In (3.37), we only wrote down a part of the Chern-Simons action of a D-brane. The complete expression is

$$S_{\mathrm{CS}} = -\frac{2\pi}{\ell_s^{p+1}}\int_{D_p}\mathrm{tr}\,\mathrm{e}^{2\pi\ell_s^2\,\mathcal{F}}\wedge\sum_p \iota^*C_p\wedge\sqrt{\frac{\hat{A}(T)}{\hat{A}(N)}}\ , \tag{5.22}$$

$\hat{A}(T)$ and $\hat{A}(N)$ are the *A-roof-genus* of the tangent and the normal bundle of the brane, respectively (see e.g. [22]). Those geometric terms are not of interest for us here, we approximate them to zeroth order as 1.

[1] We make a simplification here: In type IIB theory, the field strength F actually belongs to a U(5) gauge theory and the physical Yang-Mills field strength $\mathcal{F}$ has a component $(F^0 + T\,\iota^*B)\,T^0$ in the direction of the additional generator $T^0\sim\mathbb{1}$. This leads just to a breaking $\mathrm{U}(5)\to\mathrm{SU}(5)\times\mathrm{U}(1)$ – we are only interested in the SU(5) part and will continue with just the field strength F of SU(5). In [3], this is covered in detail.

That leaves (in addition to the $\int_{D_7} \iota^* C_8$ term which doesn't contain the Yang-Mills field strength) a term $\sim \int_{D_7} \mathrm{tr}(F) \wedge \iota^* C_6$ coupling F to C_6, a term $\sim \int_{D_7} \mathrm{tr}(F \wedge F) \wedge \iota^* C_4$ coupling F to C_4 and further terms containing C_2 and C_0. The C_6 term is immediately equal to zero as $\mathrm{tr}(F) = 0$ for $F \in \mathfrak{su}(5)$. The C_2 and C_0 terms do not lead to mass terms for non-trivial reasons, see footnote 4 in [43]. Hence we only need to consider the term

$$-\kappa \int_{D_7} \mathrm{tr}(F \wedge F) \wedge \iota^* C_4 \tag{5.23}$$

(with κ the same constant like in (5.19)).

The integral (5.23) has to be split into an integration over $\mathbb{R}^{1,3}$ and one over the GUT divisor $S = D_7 \cap X_3$. Because we still assume a product structure $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times X_3$ of spacetime like in (3.23), the field strength F can necessarily be written as a sum

$$F^a = F^a_{\mathrm{4D}}(x^i) + F^a_{\mathrm{int}}(x^I) \tag{5.24}$$

where x^i are coordinates of Minkowski space and x^I those of the internal space, and

$$F^a_{\mathrm{4D}} \in H^2(\mathbb{R}^{1,3}) \quad \text{and} \quad F^a_{\mathrm{4D}} \in H^2(S) \ . \tag{5.25}$$

The flux in external space should vanish such that only F^Y_{int} acquires a background value. (Note that F^a_{int} for $a \neq Y$ cannot get an expectation value because of the definition (5.18).)

Some notation: Let $\mathcal{C}_Y$ be the cycle in S which is Poincaré dual to $\langle F^Y_{\mathrm{int}} \rangle$ (with respect to S) and $L_Y = \mathcal{O}_S(\mathcal{C}_Y)$ the corresponding line bundle over S. Then, using once more (2.34),

$$\langle F^Y_{\mathrm{int}} \rangle = [\mathcal{C}_Y]_S = \mathrm{c}_1(\mathcal{O}_S(\mathcal{C}_Y)) = \mathrm{c}_1(L_Y) \ . \tag{5.26}$$

We can at most get a Stückelberg mass term from (5.23). We expand that action using (5.24) and see that only the cross term

$$-2\kappa \int_{D_7} \mathrm{tr}(F_{\mathrm{4D}} \wedge F_{\mathrm{int}}) \wedge \iota^* C_4$$

can give us a Stückelberg mass term like in (5.10). We now expand C_4 in terms of harmonic forms on the Calabi-Yau, this expansion is well known (see e.g. [45, Ch. 5.1] or [46, Ch. 2.3]):

$$C_4 = \sum_{i=1}^{b^{1,1}(X_3)} \left(c_2^{(i)}(x^i) \wedge \omega_i + \rho_i(x^i)\, \tilde{\omega}^i \right) + \sum_{\hat{K}=0}^{b^{1,2}(X_3)} \left(V^{\hat{K}}(x^i) \wedge \alpha_{\hat{K}} - U_{\hat{K}}(x^i) \wedge \beta^{\hat{K}} \right) \ . \tag{5.27}$$

Here $\omega_i \in H^{1,1}(X_3)$, $\tilde{\omega}^i \in H^{2,2}(X_3)$ and $\alpha_{\hat{K}}, \beta^{\hat{K}} \in H^3(X_3)$. For simplicity we ignored the fact that X_3 is actually not a manifold but an orientifold. Plugging the expansion into the action, only the $c_2^{(i)} \wedge \omega_i$ part survives and we finally see that a mass term can only arise from the term

$$-2\kappa\, \mathrm{tr}(T^a T^b) \int_{D_7} F^a_{\mathrm{4D}} \wedge \langle F^b_{\mathrm{int}} \rangle \wedge c_2^{(i)} \wedge \iota^* \omega_i = -\kappa \left(\int_S c_1(L_Y) \wedge \iota^* \omega_i \right) \int_{\mathbb{R}^{1,3}} F^Y_{\mathrm{4D}} \wedge c_2^{(i)} \ . \tag{5.28}$$

Summarizing, we need

$$\int_S \mathrm{c}_1(L_Y) \wedge \iota^* \omega = 0 \quad \forall\, \omega \in H^{1,1}(X_3) \ . \tag{5.29}$$

A Geometric Solution to the Problem. We use Poincaré duality to rewrite (5.29) to

$$0 = \int_{\mathcal{C}_Y} \iota^* \omega = \int_{\iota_* \mathcal{C}_Y} \omega \quad \forall\, \omega \in H^{1,1}(X_3) \ , \tag{5.30}$$

the second equality is just the definition A.43 of the integral. $\iota_* : H_2(S) \to H_2(X_3)$ is the *pushforward* of homology classes, defined in subsection 2.1.2. This condition is equivalent to requiring that

$$\iota_* \mathcal{C}_Y = 0 \ , \tag{5.31}$$

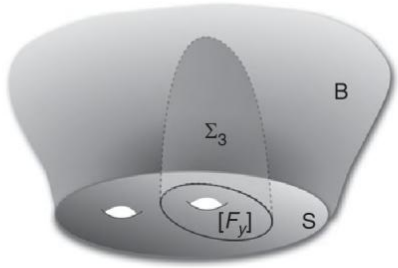


Figure 5.1: A curve $[F_Y]$ which is non-trivial in $H_2(S)$ but trivial in $H_2(B)$ because $[F_Y] = \partial\Sigma_3$. Taken from [42].

but obviously $\mathcal{C}_Y \neq 0$. Figure 5.1 shows an illustration of such a curve.

An easy way to achieve (5.31) is to take two chains $\mathcal{C}_1 \in H_2(S)$ and $\mathcal{C}_2 \in H_2(S)$ which are non-homologous in S but homologous as curves in X_3. That means that $\iota_*(\mathcal{C}_1) = \iota_*(\mathcal{C}_2)$ or, geometrically speaking, there is a 3-chain Γ in X_3 such that $\partial\Gamma = \mathcal{C}_2 - \mathcal{C}_1$. Then we just define

$$\mathcal{C}_Y = \mathcal{C}_1 - \mathcal{C}_2 \tag{5.32}$$

and we are done.

Rewriting the Condition. There is another way to write (5.30) which is used often. We define the *pushforward in cohomology*

$$\iota_! : H^2(S) \to H^4(B) \tag{5.33}$$

as the Poincaré dual of the pushforward $\iota_*\mathcal{C}_Y$ in homology. Obviously, in general $\int \Omega \wedge \iota^*\omega = \int \iota_!\Omega \wedge \omega$, therefore we can rewrite the condition to

$$\int_{X_3} \iota_!\, \mathrm{c}_1(L_Y) \wedge \omega = 0 \quad \forall\, \omega \in H^{1,1}(X_3)\ . \tag{5.34}$$

This condition is equivalent to

$$\iota_!\, \mathrm{c}_1(L_Y) = 0 \tag{5.35}$$

in $H^4(X_3)$.

5.3 Lifting Type IIB 2-form Fluxes to F-Theory 4-form Fluxes

When we compactified M-theory on an S^1 in section 4.2, we already saw how different kinds of fields in one theory can be encapsulated in just one field in a dual theory: Dimensional reduction of the M-theory A_3 form field yields the type IIA C_3 and B_2 fields, see (4.12).

The same thing happens with the fluxes: Two types of fluxes in type IIB theory, namely closed string three-form fluxes $G_3 = F_3 - \tau H_3$ and brane fluxes F_2, arise from the M-theory $(2,2)$-form flux G_4 [5,44]. Those G_4 fluxes can be constructed in quite generality, for details see [44, Ch. 5.4] or [47].

G_4-Fluxes and Brane Fluxes. We are mainly interested in the brane fluxes. In principle, they arise from G_4 by the reduction

$$G_4 = \sum_i F^{(i)} \wedge \omega_i + \cdots\ . \tag{5.36}$$

For example, for a flux along a Cartan generator of the gauge group G, ω_i is the Poincaré dual of the respective resolution divisor E_i from (4.29) (with respect to $\tilde{Y}_4$) [5].

We will elaborate how to determine the respective resolution divisor E [8, Ch. 4]: The idea is that since the resolution divisors E_i intersect like the simple roots of the Lie algebra (equation (4.29)), they correspond to the Cartan generators T^{α_i} defined in (2.57).

For example, let us say we have an SU(5) gauge group with a hypercharge flux along the curve $\mathcal{C}_Y$, like discussed above. We can use any of the resolution divisors E_i to lift $[\mathcal{C}_Y]_S$ to a 4-form flux: Fibering the $\mathbb{P}^1_i$ over the curve $\mathcal{C}_Y$ gives 4-cycles

$$\mathcal{C}_{Y,i} = \mathbb{P}^1_i \to \mathcal{C}_Y = E_i|_{\mathcal{C}_Y} \ ,$$

their Poincaré duals in $\bar{Y}_4$ are 4-form fluxes $G_{4,i}$. Since taking the exterior product in cohomology is Poincaré dual to taking the intersection in homology, we can simplify the fluxes:

$$G_{4,i} = [\mathcal{C}_{Y,i}]_{\bar{Y}_4} = \left[E_i|_{\mathcal{C}_Y} \right]_{\bar{Y}_4} = [\mathcal{C}_Y]_S \wedge [E_i]_{\bar{Y}_4} = F^Y \wedge \omega_i \ . \tag{5.37}$$

The hypercharge generator $T^Y \simeq \mathrm{diag}(-2,-2,-2,3,3)$ is the following combination of the basis (2.57) corresponding to the simple roots:

$$T^Y = -2\, T^{\alpha_1} - 4\, T^{\alpha_2} - 6\, T^{\alpha_3} - 3\, T^{\alpha_4} \ . \tag{5.38}$$

This tells us that we have to use the following 4-flux:

$$G_4 = -2\, G_{4,1} - 4\, G_{4,2} - 6\, G_{4,3} - 3\, G_{4,4} \ . \tag{5.39}$$

G_4-Fluxes in the Family W_5. The technique we have just discussed is what we will use later on. Let us mention, however, that it is also possible to first lift the flux to a four-form flux in W_E using the cylinder (4.54), and then write down a flux in any fourfold of the family W_5. Following [1], we will illustrate this technique with a simple example.

We first construct the threefold X_3 as a hypersurface in affine complex space $\mathbb{C}^4$ with coordinates z_1, z_2, z_3 and ξ:

$$X_3 : \quad \xi^2 = x_3 + 1 \ . \tag{5.40}$$

This X_3 is not a Calabi-Yau manifold and not even compact, but it will suffice for this example.

X_3 is a double cover of the base $B_3 = \mathbb{C}^3_{(x_1,x_2,x_3)}$ like in section 4.5, we can read off $b_2 = x_3 + 1$. The orientifold plane is located at the branch point $b_2 = 0$, i.e. $x_3 = -1$ in B_3. We want to put an SU(2) stack of branes on the divisor $S : x_3 = 0$. Looking this up in table 4.2, we can achieve it with an I_2^s singularity

$$b_2 = x_3 + 1 \ , \quad b_4 = x_3 \ , \quad b_6 = 0 \ . \tag{5.41}$$

Plugging (5.41) back into the defining equation of the conic bundle W_E (4.52) and rewriting that equation, we get

$$W_E : \quad (y+s)(y-s) = x_3 s\,(s + 2t) \ . \tag{5.42}$$

This has still a singularity at $s = y = x_3 = 0$. We can resolve the singularity after introducing an auxiliary coordinate σ with the equation $x_3 = \sigma$ such that W_E is given by

$$W_E : \quad \begin{cases} (y+s)(y-s) = \sigma s\,(s+2t) \\ x_3 = \sigma \end{cases} \ ,$$

where the coordinates of the fiber have the scaling relation

$$\begin{array}{cccc} s & y & t & \sigma \\ \hline 1 & 1 & 1 & 0 \end{array} \ .$$

We know how to blow such a space up in $s = y = \sigma = 0$: We introduce a new coordinate v with scaling relations

$$\begin{array}{ccccc} s & y & t & \sigma & v \\ \hline 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & -1 \end{array}$$

and SR ideal $\langle syt, sy\sigma, vt\rangle$. The resolved conic bundle is given by the equations

$$\tilde{W}_E : \quad \begin{cases} (y+s)(y-s) = \sigma s\,(vs+2t) \\ x_3 = \sigma v \end{cases} . \tag{5.43}$$

The cylinder $R_3 = \tilde{W}_E \cap \{\Delta_E = 0\} = \tilde{W}_E \cap \{x_3 = 0\}$ now has three components

$$R_\sigma^+ : \quad \begin{cases} x_3 = 0 \\ \sigma = 0 \\ y+s = 0 \end{cases} \qquad R_\sigma^- : \quad \begin{cases} x_3 = 0 \\ \sigma = 0 \\ y-s = 0 \end{cases} \qquad R_v : \quad \begin{cases} x_3 = 0 \\ v = 0 \\ y^2 - s^2 = 2t\sigma s \end{cases} , \tag{5.44}$$

each fiber consisting of three $\mathbb{P}^1$s intersecting like $\mathbb{P}_\sigma^+ \cup_{p_+} \mathbb{P}_v \cup_{p_-} \mathbb{P}_\sigma^-$ in the points $p_\pm : x_3 = v = \sigma = y \pm s = 0$. $\mathbb{P}_\sigma^+$ and $\mathbb{P}_\sigma^-$ do not intersect since $sy\sigma$ is in the SR ideal.

Let us now consider fluxes F_1 and F_2 along the two branes. We can lift them to $\tilde{W}_E$ in a similar fashion to the one described above, by fibering the exceptional $\mathbb{P}^1$s over the dual 2-cycles. As discussed, this is equivalent to taking the exterior product of F_i with the two-form dual to the cylinder component. It turns out that, due to the requirement of Poincaré invariance, the correct linear combinations are

$$G_{4,1} = \frac{1}{2} F_1 \wedge ([R_\sigma^+] - [R_\sigma^-] - [R_v])$$
$$G_{4,2} = \frac{1}{2} F_2 \wedge ([R_\sigma^+] + [R_v] - [R_\sigma^-]) .$$

The Poincaré duals are understood with respect to $\tilde{W}_E$. For a Cartan flux with $F_2 = F = -F_1$, the result is

$$G_4 = F \wedge [R_v] . \tag{5.45}$$

Now that we have lifted the two-form flux F to a four-form G_4 in $\tilde{W}_E$, we'd usually have to put in a bit more work in order to define a flux in a four-fold of the family $\tilde{W}_5$. In this case it is easy, however, because we can write

$$G_4 = F \wedge [E] \tag{5.46}$$

with the exceptional divisor $E = \{v = 0\}$, this is well defined in a generic four-fold of the family $\tilde{W}_5$.

5.4 A First SU(2) Example

We already constructed a model giving an SU(2) gauge group in section 5.3, but it didn't satisfy all of our requirements. In the following, we will construct another such model with a compact Calabi-Yau Y_4, where the D-brane divisor is rigid[2]. This example is taken from [1].

The Ambient Variety T_4. We start from a $\mathbb{P}^4$ with homogeneous coordinates $z_0, \dots, z_4$ and SR ideal $\langle z_0 \cdots z_4\rangle$. Blowing this up in $(0 : \cdots : 0 : 1)$ gives a toric variety T_4 described by the coordinates

$$\begin{array}{cccccc} z_0 & z_1 & z_2 & z_3 & z_4 & w \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \tag{5.47}$$

and the SR ideal $\langle z_0 z_1 z_2 z_3, z_4 w\rangle$. This variety has two divisor classes $[z]$ and $[w]$, corresponding to the two rows, and $[z_0] = \cdots = [z_3] = [z]$ and $[z_4] = [z] + [w]$. Since the wedge product of cohomology classes corresponds to intersection in homology, the SR ideal tells us that[3]

$$[z]^4 = 0 \quad \text{and} \quad [w]\,([z] + [w]) = 0 \tag{5.48}$$

[2] A Calabi-Yau 3-fold is rigid if it has no complex structure moduli, i.e. $b^{2,1} = 0$.

[3] In this context, the exterior product $[z_1] \wedge [z_2]$ is usually written as multiplication $[z]^2$. It is also commutative since we are only using forms of even degree.

as discussed in subsection 2.2.4.

The product of four classes is dual to the intersection of four hypersurfaces, consisting only of discrete points. The number of points in the intersection is called the *intersection number*. By Poincaré duality, it can be calculated as

$$\#([z]^n[w]^{4-n}) = \int_{T_4} [z]^n[w]^{4-n} \ . \tag{5.49}$$

In $\mathbb{P}^4$, there is only one class $[z]_{\mathbb{P}^4}$ and the normalized volume form is $\mathrm{vol}_{\mathbb{P}^4} = [z]^4_{\mathbb{P}^4}$ meaning that $\int_{\mathbb{P}^4} \mathrm{vol}_{\mathbb{P}^4} = 1$. Let $\sigma : T_4 \to \mathbb{P}^4$ be the blow-up map, then the volume form vol of the blow-up T_4 with $\int_{T_4} \mathrm{vol} = 1$ is given by $\mathrm{vol} = \sigma^* \mathrm{vol}_{\mathbb{P}^4}$ because

$$\int_{T_4} \sigma^* \mathrm{vol}_{\mathbb{P}^4} = \int_{\sigma_* T_4 = \mathbb{P}^4} \mathrm{vol}_{\mathbb{P}^4} = 1 \ .$$

This can be explicitly calculated:

$$\mathrm{vol} = (\sigma^*[z]_{\mathbb{P}^4})^4 = ([z]+[w])^4 = [z]^4 + 4\,[z]^3[w] + 6\,[z]^2[w]^2 + 4\,[z][w]^3 + [w]^4 = -[w]^4 \ . \tag{5.50}$$

The pullback of $[z]_{\mathbb{P}^4}$ is $[z]+[w]$ because the pre-image of such a hypersurface in $\mathbb{P}^4$ is either of the type $\{z_4 = 0\}$ or of the type $\{z_0 = 0\} \cup \{w = 0\}$, both are Poincaré dual to $[z]+[w]$.

All of this shows us that $\#([w]^4) = -1$ and using (5.48) we get the *intersection ring*

$$\begin{array}{cccccc} & [z]^4 & [z]^3[w] & [z]^2[w]^2 & [z][w]^3 & [w]^4 \\ \# & 0 & 1 & -1 & 1 & -1 \end{array} \ . \tag{5.51}$$

The Base B_3. We choose to take B_3 to be a hypersurface in T_4 with class $[B_3] = 3\,[z] + [w]$. In other words, it is given by a polynomial of degree $(3,1)$, such a polynomial has the form

$$B_3 : \quad P_2(z_0,\dots,z_3)\,z_4 + P_3(z_0,\dots,z_3)\,w = 0 \tag{5.52}$$

in general (the polynomials P_i are of degree i).

We are interested in the cohomology of B_3. In fact, the class $[B_3]$ was chosen like this because that makes $\mathcal{O}(B_3)$ *ample* and we can use the Lefshetz hyperplane theorem (see subsection 2.2.2 or theorem C.30). Ampleness can easily be checked with a computer algebra system, see section D.2. The hyperplane theorem directly shows that $i^*[z]$ and $i^*[w]$ are a basis for $H^2(B_3)$ if $i : B_3 \to T_4$ is the embedding. We will mostly omit the i^* in the following.

The intersection ring on B_3 can be easily calculated, for example $\int_{B_3} i^*[z]^3 = \int_{i_* B_3} [z]^3 = \int_{T_4} [B_3] \wedge [z]^3 = 3\int_{T_4} [z]^4 + \int_{T_4} [z]^3[w] = 1$. The result is

$$\begin{array}{ccccc} & [z]^3 & [z]^2[w] & [z][w]^2 & [w]^3 \\ \# & 1 & 2 & -2 & 2 \end{array} \ . \tag{5.53}$$

Finally, we are interested in the first Chern class of B_3. We begin our calculation with remembering $K_{\mathbb{P}^4} = \mathcal{O}(-5)$ from (2.37). Using (2.47) we find that

$$K_{T_4} = \sigma^* \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \mathcal{O}(3[w]) \tag{5.54}$$

with first Chern class $\mathrm{c}_1(T_4) = -\mathrm{c}_1(K_{T_4}) = 5\,\sigma^*[z]_{\mathbb{P}^4} - 3\,[w] = 5\,[z] + 2\,[w]$. This confirms the claim made in subsection 2.2.4: In toric varieties, we generally only have to sum the entries of the scaling relation table in order to read off the first Chern class. Using the adjunction formula (2.33), we get

$$\mathrm{c}_1(B_3) = i^*\big(\mathrm{c}_1(T_4) - \mathrm{c}_1(\mathcal{O}(B_3))\big) = 5\,[z] + 2\,[w] - [B_3] = 2\,[z] + [w] \ . \tag{5.55}$$

Placing the Branes. We choose

$$S = B_3 \cap \{w = 0\} \tag{5.56}$$

as the divisor of B_3 on which we will put the D-branes. When w is set to zero, we can set $z_4 = 1$ due to the SR ideal. The defining equations of S are thus

$$S: \quad w = 0 \quad \text{and} \quad P_2(z_0, \ldots, z_3) = 0 \tag{5.57}$$

in T_4, this is a rigid surface in B_3 [1].

S is given as a quadric $P_2 = 0$ in the $\mathbb{P}^3$ spanned by z_0, z_1, z_2 and z_3. Any smooth quadric surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, we will prove this quickly [48, Ch. 4.1]: The Segré map

$$\big((s_0 : s_1), (t_0 : t_1)\big) \mapsto (z_0 : z_1 : z_2 : z_3) = (s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1) \tag{5.58}$$

is an embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ with the image given by $z_0z_3 - z_1z_2 = 0$ – a quadric in $\mathbb{P}^3$. According to Bertini's theorem C.37, all smooth quadric surfaces in $\mathbb{P}^3$ look the same.

Given our quadric $P_2 = 0$ we can now use the Segré map (5.58) to redefine the coordinates z_0 to z_3 such that

$$P_2(z_0, \ldots, z_3) = z_0z_3 - z_1z_2 \tag{5.59}$$

and the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the Segré map.

We want to write down a fibration with an I_2^s singularity along the divisor $S : w = 0$ in B_3, giving an SU(2) gauge group like in the previous section. According to table 4.2, b_4 and b_6 need to scale as w and w^2, respectively – we will put $b_4 = b_{4,1}w$ and $b_6 = b_{6,2}w^2$ – and $b_2 = a_1^2 + a_{2,1}w$. In order to describe the F-theory 4-fold Y_4 using the definition of the base (5.52) and the Weierstrass equation (4.19), we consider the bundle $\mathbb{P}^2_{(2:3:1)} \to W_6 \to T_4$ with s, y and u being the fiber coordinates[4]. We know that they have to transform as the sections of certain powers of the anticanonical bundle over B_3, for example s is a section of $K_{B_n}^{-2} = 2\,\mathcal{O}(2\,[z] + [w])$.

We summarize that bundle using the table

$$W_6: \quad \begin{array}{ccccccccc} z_0 & z_1 & z_2 & z_3 & z_4 & w & s & y & u \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 \end{array} \tag{5.60}$$

with the SR ideal generated by $\langle z_0z_1z_2z_3, z_4w, syu\rangle$. Y_4 is given as a complete intersection in W_6:

$$Y_4: \quad \begin{cases} (z_0z_3 - z_1z_2)\, z_4 + P_3(z_0, \ldots, z_3)\, w = 0 \\ y^2 = s^3 + (a_1^2 + a_{2,1}w)\, s^2u^2 + 2b_{4,1}wsu^4 + b_{6,2}w^2u^6 \end{cases} . \tag{5.61}$$

Note that (by construction)

$$\mathrm{c}_1(Y_4) = (15\,[z] + 7\,[w] + 6\,[u]) - (3\,[z] + [w]) - (12\,[z] + 6\,[w] + 6\,[u]) = 0\,.$$

Turning on a Cartan Flux. As discussed in section 5.2, we are now looking for two 2-cycles $\mathcal{C}_1$ and $\mathcal{C}_2$ that are homologous in $S = \mathbb{P}^1_s \times \mathbb{P}^1_t$ but non-homologous in B_3. Then we can turn on a Cartan flux Poincaré dual to $\mathcal{C}_F = \mathcal{C}_1 - \mathcal{C}_2$ and lift it to F-theory.

We claim that the following cycles satisfy that condition:

$$\mathcal{C}_1 = T_4 \cap \{z_0 = z_1 = w = 0\} = S \cap \{s_0 = 0\} \text{ and} \tag{5.62}$$
$$\mathcal{C}_2 = T_4 \cap \{z_0 = z_2 = w = 0\} = S \cap \{t_0 = 0\}\,. \tag{5.63}$$

[4] Where we use the name "u" instead of "z" to avoid confusion, and $s = x - \frac{1}{3}b_2u^2$ like in section 4.5.

- Let's first prove that they are non-homologous in S. As we know, S is isomorphic to $\mathbb{P}^1_s \times \mathbb{P}^1_t$ with coordinates

s_0	s_1	t_0	t_1
1	1	0	0
0	0	1	1

  and H^2 spanned by the divisor classes $[s]$ and $[t]$.

  $\mathcal{C}_1$ as a cycle of S is dual to $[s]$ and $\mathcal{C}_2$ is dual to $[t]$, they are not equal. More explicitly we can even show that

$$\int_S [\mathcal{C}_F]^2 = \int_{\mathbb{P}^1_s \times \mathbb{P}^1_t} ([s]-[t])^2 = -2 \left(\int_{\mathbb{P}^1_s} [s]\right)\left(\int_{\mathbb{P}^1_t} [t]\right) = -2 \neq 0 \, .$$

- In T_4 on the other hand, the cycles are both dual to the class

$$[z]^2[w] \in H^6(T_4)$$

  and therefore homologous. The Lefshetz hyperplane theorem finally shows that the cycles are also homologous in B_3, that concludes the proof.

Lifting the Cartan Flux to F-Theory. We continue like in section 5.3 and resolve the singularity at $s = y = w = 0$ in Y_4. Blowing up the 6-fold in that point gives

$$\tilde{W}_6 : \quad \begin{array}{cccccccccc} z_0 & z_1 & z_2 & z_3 & z_4 & w & s & y & u & v \\ \hline 1 & 1 & 1 & 1 & 1 & 0 & 4 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 \end{array} \, . \tag{5.64}$$

the SR ideal is generated by the old generators $\langle z_0z_1z_2z_3, z_4w, syu\rangle$, by the previously singular $\langle syw\rangle$ and finally by $\langle z_4v, tv\rangle$ making the blow-up map

$$(z_0 : z_1 : z_2 : z_3 : z_4 : w : s : y : t : v) \mapsto (z_0 : z_1 : z_2 : z_3 : z_4 : wv : sv : yv : t)$$

well-defined. The resolved $\tilde{Y}_4$ is a total intersection in $\tilde{W}_6$, namely

$$\tilde{Y}_4 : \quad \begin{cases} (z_0z_3 - z_1z_2)\, z_4 + P_3(z_0, \ldots, z_3)\, wv = 0 \\ y^2 = s^3 v + (a_1^2 + a_{2,1}wv)\, s^2u^2 + 2b_{4,1}wsu^4 + b_{6,2}w^2u^6 \end{cases} \, . \tag{5.65}$$

We already know that we have to lift the curves along the exceptional divisor if the flux is along the Cartan of SU(2): We restrict the exceptional divisor $v = 0$ to the curves $\mathcal{C}_1 = \{z_0 = z_1 = vw = 0\}$ and $\mathcal{C}_2 = \{z_0 = z_2 = vw = 0\}$. This gives the H_4-cycles

$$\Theta_1 : \quad \begin{cases} v = 0 \\ z_0 = 0 \\ z_1 = 0 \\ y^2 = a_1^2 s^2 u^2 + 2b_{4,1}wsu^4 + b_{6,2}w^2u^6 \end{cases} \qquad \Theta_2 : \quad \begin{cases} v = 0 \\ z_0 = 0 \\ z_2 = 0 \\ y^2 = a_1^2 s^2 u^2 + 2b_{4,1}wsu^4 + b_{6,2}w^2u^6 \end{cases}$$

in $\tilde{T}_6$ (lying in $\tilde{Y}_4$) and the Cartan flux is

$$G_4 = [\Theta_1 - \Theta_2]_{\tilde{Y}_4} \, . \tag{5.66}$$

Chapter 6

Constructing a Realistic Model

We continue with a more realistic model, where an SU(5) GUT is broken to the MSSM using a hypercharge flux. Also this example was taken from [1].

6.1 Geometry of the 4-fold

Setting the Stage. We start with a toric variety T_4 described by the coordinates

$$T_4 : \quad \begin{array}{cccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ \hline 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \tag{6.1}$$

with SR ideal $\langle z_1z_2z_3, z_4z_5z_6\rangle$. The base manifold B_3 is defined as the vanishing locus of a degree $(5,2)$ polynomial which has the following form in general:

$$\begin{aligned} B_3 : \quad P_{(5,2)}(z_1,\ldots,z_6) &= z_4^2\,P_1 + z_4z_5\,R_3 + z_4z_6\,P_2 + z_5^2\,Q_5 + z_5z_6\,R_4 + z_6^2P_3 \\ &= z_5\,(z_5\,Q_5 + z_4\,R_3 + z_6\,R_4) + \left[z_4^2\,P_1 + z_4z_6\,P_2 + z_6^2P_3\right] = 0\;. \end{aligned} \tag{6.2}$$

P_i, Q_i and R_i are polynomials in (z_1,z_2,z_3) of degree i.

The surface S on which we will put the branes is

$$S : \quad B_3 \cap \{z_5 = 0\}\;. \tag{6.3}$$

By setting $z_5 = 0$ in T_4 we see that we can alternatively describe S as the vanishing locus of

$$S : \quad z_4^2\,P_1 + z_4z_6\,P_2 + z_6^2\,P_3 = 0 \tag{6.4}$$

in the ambient space

$$T_3 : \quad \begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_6 \\ \hline 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} = \begin{array}{ccccc} z_1 & z_2 & z_3 & z_4 & z_6 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\;. \tag{6.5}$$

with SR ideal $\langle z_1z_2z_3, z_4z_6\rangle$. We recognize T_3 to be the blow-up of the $\mathbb{P}^3$ spanned by $z_1,\ldots,z_4$ in a point. Furthermore, S arises as the blow-up of a cubic $z_4^2P_1 + z_4P_2 + P_3 = 0$ in that $\mathbb{P}^3$. This shows that S is a *del Pezzo surface*:

Del Pezzo surfaces are surfaces X with ample anticanonical bundle $\bar{K}_X$. The *degree* of a del Pezzo surface is defined as the self-intersection number $\mathcal{K}_X^2 = \int_X \mathrm{c}_1(K_X)^2$ of the canonical class $\mathcal{K}_X = \mathrm{c}_1(K_X)$. A classical theorem [49, Cor. V.4.7] states that the only del Pezzo surfaces are $\mathbb{P}^1\times\mathbb{P}^1$ of degree 8 and dP_n, the blow-up of $\mathbb{P}^2$ in $n \in \{0,1,\ldots,8\}$ generic points, of degree $(9-n)$. Only $\mathbb{P}^1\times\mathbb{P}^1$ and dP_0 through dP_3 are toric varieties.

S is a dP_7 surface since, generically, a cubic in $\mathbb{P}^3$ is a dP_6 surface [49].

Non-Trivial Curve on S which is Trivial on B_3. We need to find curves which are homologous in B_3 but non-homologous in S. Remember our previous example (section 5.4): S was given as a quadric in $\mathbb{P}^3$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, a del Pezzo surface of degree 8. By appropriately restricting the defining equation to (5.59) we were able to figure out that the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the requirement.

In the present case, $S = \mathrm{dP}_7$ is given by a polynomial of degree $(5, 2)$ in the ambient space T_3. We restrict the defining equation to

$$Q_{(5,2)} = z_1 z_2 z_6 \, F_1(z_1 z_6, z_2 z_6, z_3 z_6, z_4) + z_3 z_4 \, \tilde{F}_1(z_1 z_6, z_2 z_6, z_3 z_6, z_4) = 0 \tag{6.6}$$

(where F_1 and $\tilde{F}_1$ are linear combinations of their arguments) and define the curves

$$\begin{aligned} \mathcal{C}_{13}: &\quad \{z_1 = z_3 = z_5 = 0\} \,, \\ \mathcal{C}_{24}: &\quad \{z_2 = z_4 = z_5 = 0\} \text{ and} \\ \mathcal{C}_{63}: &\quad \{z_6 = z_3 = z_5 = 0\} \,. \end{aligned} \tag{6.7}$$

Those curves lie in S because of how we restricted the defining equation.

The curves satisfy the relation $\mathcal{C}_{13} + \mathcal{C}_{63} = \mathcal{C}_{24}$ in T_4 (because $[z_1][z_3][z_5] + [z_6][z_3][z_5] = [z_2][z_4][z_5]$). This relation also holds on B_3 because $\mathcal{O}(B_3)$ is ample. This shows that the curve

$$\mathcal{C}_Y = \mathcal{C}_{24} - \mathcal{C}_{13} - \mathcal{C}_{63} \tag{6.8}$$

is trivial in B_3, we claim that it is non-trivial in S. To prove this, we will calculate the self-intersection product $\mathcal{C}_Y \cdot \mathcal{C}_Y$ in S and show that it is non-zero.

In general, for a curve C in a surface S,

$$2g_C - 2 = C \cdot (C + \mathcal{K}_S) \,,$$

where g_C is the genus of the curve. This follows from the adjunction formula, see [49, Prop. V.1.5]. In our case, $S = \mathrm{dP}_7$ and it is clear from the definition that all 2-cycles of a dP_n are $\mathbb{P}^1$s with genus $g_{\mathbb{P}^1} = 0$. Thus,

$$C^2 = -2 - C \cdot \mathcal{K}_S \,.$$

Next, we will calculate $\mathcal{K}_S$ using the adjunction formula (2.33). We see that

$$\mathcal{K}_S = \left(\mathcal{K}_{T_3} + \mathrm{c}_1(\mathcal{O}(S))\right)\big|_S = \left(-4\,[z_1] - 2\,[z_6] + 3\,[z_1] + 2\,[z_6]\right)\big|_S = -\left.[z_1]\right|_S \,.$$

Finally, using Poincaré duality we can calculate

$$C^2 = -2 + \int_S [C]_S \wedge \left.[z_1]\right|_S = -2 + \int_C \left.[z_1]\right|_C = -2 + \int_{T_3} [C] \wedge [z_1]$$

(where the Poincaré duals are with respect to T_3 in the last term).

Because $[z_1]^3 = 0$ as well as $([z_1] + [z_6]) \wedge [z_6] = 0$ (from the SR ideal), and because the volume form on T_3 is $([z_1] + [z_6])^3$, the intersection ring of T_3 is

$$\begin{array}{ccccc} & [z_1]^3 & [z_1]^2[z_6] & [z_1][z_6]^2 & [z_6]^3 \\ \# & 0 & 1 & -1 & 1 \end{array} \,.$$

With this knowledge, we can directly calculate

$$\begin{aligned} \mathcal{C}_{13}^2 &= -2 + \int [z_1]^3 = -2 \,, \\ \mathcal{C}_{24}^2 &= -2 + \int [z_1]^2 \wedge ([z_1] + [z_6]) = -1 \text{ and} \\ \mathcal{C}_{63}^2 &= -2 + \int [z_1]^2 [z_6] = -1 \,. \end{aligned}$$

Also, we can easily read off the other intersection numbers,

$$\mathcal{C}_{13} \cdot \mathcal{C}_{24} = 0 \,, \quad \mathcal{C}_{13} \cdot \mathcal{C}_{63} = 1 \quad \text{and} \quad \mathcal{C}_{24} \cdot \mathcal{C}_{63} = 0$$

from the definition of T_3. This proves that

$$\mathcal{C}_Y^2 = (\mathcal{C}_{24} - \mathcal{C}_{13} - \mathcal{C}_{63})^2 = -2 \neq 0 \,. \tag{6.9}$$

Placing the Branes. We let ourselves be guided by the example given in section 4.4 on how to model an SU(5) gauge group in F-theory. We make the ansatz (4.32) with the additional constraint $\mathfrak{b}_0 = 0$ ("U(1)-restricted Tate model"). This simplifies the expression S_1 describing the I_1 singularity along which the symmetry is enhanced and gives us an additional massless $\mathrm{U}(1)_X$ symmetry which does *not* arise as the Cartan of a non-abelian gauge group [29]. (This was first described in [50].) Summarized:

$$a_1 = \mathfrak{b}_5, \quad a_2 = \mathfrak{b}_4\, z_5, \quad a_3 = \mathfrak{b}_3\, z_5^2, \quad a_4 = \mathfrak{b}_2\, z_5^3, \quad a_6 = 0\,, \tag{6.10}$$

$$\Delta = z_5^5 \cdot \underbrace{\left(\mathfrak{b}_5^4 P + Q\, z_5 + R'\, z_5^2 + S\, z_5^3 + T\, z_5^4\right)}_{S_1}\,. \tag{6.11}$$

P, Q, R', S and T are polynomials[1] in the $\mathfrak{b}_i$, we quote from [29]:

$$\begin{aligned}
P &= \mathfrak{b}_3\left(\mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5\right)\,, \\
Q &= \mathfrak{b}_5^2\left(8\,\mathfrak{b}_3^2\mathfrak{b}_4^2 - \mathfrak{b}_2^2\mathfrak{b}_5^2 - 8\,\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4\mathfrak{b}_5 - \mathfrak{b}_3^3\mathfrak{b}_5\right)\,, \\
R' &= 16\,\mathfrak{b}_3^2\mathfrak{b}_4^3 - 36\,\mathfrak{b}_3^3\mathfrak{b}_4\mathfrak{b}_5 - 16\,\mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_4^2\mathfrak{b}_5 + 30\mathfrak{b}_2\mathfrak{b}_3^2\mathfrak{b}_5^2 - 8\,\mathfrak{b}_2^2\mathfrak{b}_4\mathfrak{b}_5^2\,, \\
S &= 96\,\mathfrak{b}_2^2\mathfrak{b}_3\mathfrak{b}_5 - 16\,\mathfrak{b}_2^2\mathfrak{b}_4^2 - 72\,\mathfrak{b}_2\mathfrak{b}_3^3\mathfrak{b}_4 + 27\,\mathfrak{b}_3^4\,, \\
T &= 64\,\mathfrak{b}_2^3\,.
\end{aligned}$$

The resulting elliptically fibered Y_4 will therefore have a split SU(5) singularity along $S = \{z_5 = 0\}$ and a conifold singularity along the vanishing locus of S_1, which is

$$S_1: \quad \mathfrak{b}_2 = \mathfrak{b}_3 = 0 \tag{6.12}$$

as we can see. Y_4 is given by the Weierstraß equation in Tate form (4.30),

$$y^2 = x^3 + \mathfrak{b}_5\, xyz + \mathfrak{b}_4\, z_5\, x^2 z^2 + \mathfrak{b}_3\, z_5^2\, yz^3 + \mathfrak{b}_2\, z_5^3\, xz^4\,, \tag{6.13}$$

together with (6.2) in an ambient six-fold T_6 which is $\mathbb{P}^2_{(2:3:1)}$ fibered over T_4:

$$T_6: \quad \begin{array}{ccccccccc}
z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & x & y & z \\ \hline
1 & 1 & 1 & 2 & 0 & 1 & 2 & 3 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1
\end{array}\,. \tag{6.14}$$

The SR ideal is $\langle z_1 z_2 z_3, z_4 z_5 z_6, xyz\rangle$.

6.2 Singularity Resolution

Equation (6.13) is highly singular, we will perform a series of 5 blow-ups to resolve the singularities. A similar calculation can be found in [51, Ch. 2.2].

Note that there are different *blow-up routes*, we can perform the blow-ups in different order. Of course, each route leads to the same scaling relation, but the SR ideals can be different. In fact, this can lead to six different Calabi-Yau 4-folds, corresponding to six different triangulations of the underlying polytope of the toric variety. This is discussed in detail in [29]. Here we will simply describe a route leading to the result in [1].

First Blow-Up: The singularity $x = y = z_5 = 0$ is immediately obvious from (6.13) (because all terms are at least quadratic in these variables). We perform the blow-up introducing a new coordinate v_1 and write the defining equations (6.2) and (6.13) as

$$Y_4^{(1)}: \quad \begin{cases} 0 = v_1 z_5\left(v_1 z_5\, Q_5 + z_4\, R_3 + z_6\, R_4\right) + Q_{(5,2)} \\ y^2 = v_1 x^3 + \mathfrak{b}_5\, xyz + \mathfrak{b}_4\, z_5\, v_1 x^2 z^2 + \mathfrak{b}_3\, z_5^2\, v_1 yz^3 + \mathfrak{b}_2\, z_5^3\, v_1^2 xz^4 \end{cases}$$

[1] This definition ensures that the polynomials P here and in (4.33) are equal. R and R' are different, though.

in terms of the new coordinates of

$$T_6^{(1)}: \quad \begin{array}{cccccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & x & y & z & v_1 \\ \hline 1 & 1 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \end{array}$$

with SR ideal $\langle 123, 456, xyz, xy5, z\bar{1}, 46\bar{1}\rangle$. To improve readability we have written k instead of z_k and $\bar{k}$ instead of v_k.

Second Blow-Up: This new equation has an obvious singularity at $x = y = v_1 = 0$. By the same procedure,

$$Y_4^{(2)}: \quad \begin{cases} 0 = v_1 v_2 z_5 \left(v_1 v_2 z_5\, Q_5 + z_4\, R_3 + z_6\, R_4\right) + Q_{(5,2)} \\ y^2 = v_1 v_2^2 x^3 + \mathfrak{b}_5\, xyz + \mathfrak{b}_4\, z_5\, v_1 v_2 x^2 z^2 + \mathfrak{b}_3\, z_5^2\, v_1 y z^3 + \mathfrak{b}_2\, z_5^3\, v_1^2 v_2 x z^4 \end{cases}$$

in the ambient toric variety

$$T_6^{(2)}: \quad \begin{array}{ccccccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & x & y & z & v_1 & v_2 \\ \hline 1 & 1 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \end{array}$$

with SR ideal $\langle 123, 456, xyz, xy5, z\bar{1}, 46\bar{1}, xy\bar{1}, z\bar{2}, 5\bar{2}, 46\bar{2}\rangle$.

There are no obvious singularities any more. Let us rewrite the Weierstrass equation in the following form:

$$y\left(y - \mathfrak{b}_3\, z_5^2\, v_1 z^3 - \mathfrak{b}_5\, xz\right) = x v_1 v_2 \left(v_2 x^2 + \mathfrak{b}_4\, z_5\, xz^2 + \mathfrak{b}_2\, z_5^3\, v_1 z^4\right) \; . \tag{6.15}$$

This is a so-called *binomial equation* [52, Ch. 5]. It has codimension 2 singularities at the loci where one of the factors on each side is zero. We have to perform blow-ups in $y = v_1 = 0$, $y = v_2 = 0$ and $y = x = 0$. The blow-up of $y = x = 0$ resolves the conifold singularity.

Third Blow-Up: The next blow-up we do is the one in $y = v_1 = 0$, we'll call the new coordinate v_4. What we get is

$$Y_4^{(3)}: \quad \begin{cases} 0 = v_1 v_2 v_4 z_5 \left(v_1 v_2 v_4 z_5\, Q_5 + z_4\, R_3 + z_6\, R_4\right) + Q_{(5,2)} \\ v_4 y^2 = v_1 v_2^2 x^3 + \mathfrak{b}_5\, xyz + \mathfrak{b}_4\, z_5\, v_1 v_2 x^2 z^2 + \mathfrak{b}_3\, z_5^2\, v_1 v_4 y z^3 + \mathfrak{b}_2\, z_5^3\, v_1^2 v_2 v_4 x z^4 \end{cases}$$

in the ambient toric variety

$$T_6^{(3)}: \quad \begin{array}{cccccccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & x & y & z & v_1 & v_2 & v_4 \\ \hline 1 & 1 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \end{array} \; .$$

We subtracted the last line of the table from the second-to last in accordance with [1]. The SR ideal is $\langle 123, 456, xyz, xy5, z\bar{1}, 46\bar{1}, z\bar{2}, 5\bar{2}, 46\bar{2}, y\bar{1}, z\bar{4}, 46\bar{4}, x\bar{4}\rangle$.

Fourth Blow-Up: At this point we will resolve the conifold singularity by blowing up $x = y = 0$. The new coordinate will be called ℓ:

$$Y_4^{(4)}: \quad \begin{cases} 0 = v_1 v_2 v_4 z_5 \left(v_1 v_2 v_4 z_5\, Q_5 + z_4\, R_3 + z_6\, R_4\right) + Q_{(5,2)} \\ v_4 \ell y^2 = v_1 v_2^2 \ell^2 x^3 + \mathfrak{b}_5\, \ell xyz + \mathfrak{b}_4\, z_5\, v_1 v_2 \ell x^2 z^2 + \mathfrak{b}_3\, z_5^2\, v_1 v_4 y z^3 + \mathfrak{b}_2\, z_5^3\, v_1^2 v_2 v_4 x z^4 \end{cases}$$

$$\text{in } T_6^{(4)}: \quad \begin{array}{ccccccccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & x & y & z & v_1 & v_2 & v_4 & \ell \\ \hline 1 & 1 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \end{array}$$

with SR ideal $\langle 123, 456, z\bar{1}, 46\bar{1}, z\bar{2}, 5\bar{2}, 46\bar{2}, y\bar{1}, z\bar{4}, 46\bar{4}, x\bar{4}, xy, z\ell, 5\ell, \bar{1}\ell, \bar{4}\ell\rangle$.

Final Blow-Up: Finally, it remains to resolve the singularity $y = v_2 = 0$. The result is given by the equations

$$\tilde{Y}_4: \quad \begin{cases} 0 = v_1 v_2 v_3 v_4 z_5 \left(v_1 v_2 v_3 v_4 z_5\, Q_5 + z_4\, R_3 + z_6\, R_4\right) + Q_{(5,2)} \\ v_3 v_4 \ell y^2 = v_1 v_2^2 v_3 \ell^2 x^3 + \mathfrak{b}_5\, \ell xyz + \mathfrak{b}_4\, z_5\, v_1 v_2 \ell x^2 z^2 + \mathfrak{b}_3\, z_5^2\, v_1 v_4 y z^3 + \mathfrak{b}_2\, z_5^3\, v_1^2 v_2 v_4 x z^4 \end{cases} \tag{6.16}$$

in the ambient toric variety $\tilde{T}_6$ described by

$$\tilde{T}_6: \quad \begin{array}{cccccccccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & x & y & z & v_1 & v_2 & v_3 & v_4 & \ell \\ \hline 1 & 1 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{array} \tag{6.17}$$

with SR ideal

$$\langle 123, 456, 46\bar{1}, 46\bar{2}, 46\bar{3}, 46\bar{4}, xy, z\bar{1}, z\bar{2}, z\bar{3}, z\bar{4}, z\ell, y\bar{1}, y\bar{2}, x\bar{3}, x\bar{4}, 5\bar{2}, 5\bar{3}, 5\ell, \bar{1}\ell, \bar{4}\ell, \bar{1}\bar{3}\rangle\ . \tag{6.18}$$

6.3 Fiber Structure

We are interested in the fiber structure over the brane, that is in the fiber structure of the divisor

$$z_5 v_1 v_2 v_3 v_4 = 0 \tag{6.19}$$

in $\tilde{Y}_4$.[2] For now, we will consider a generic point of the brane, where none of the $\mathfrak{b}_i$ vanish.

We claim that each fiber has 5 irreducible components $\mathbb{P}^1_0, \ldots, \mathbb{P}^1_4$, where $\mathbb{P}^1_0 = \{z_5 = 0\}$ and $\mathbb{P}^1_i = \{v_i = 0\}$ over a point in $\tilde{S}$. Those fibers intersect like the extended Dynkin diagram of SU(5) (figure 6.1), see also [29, App. A]. In the following, we will clarify what we mean by that, and prove these statements.

[2] We made a short-cut in our notation compared to the one used in section 5.3: There we introduced the auxiliary coordinate σ together with the equation $x_3 = \sigma$, in the present context that would be $z_5 = \sigma$. If we had done this, we would get (6.16) with all z_5 replaced by σ together with the third equation $z_5 = \sigma v_1 v_2 v_3 v_4$. This explains why the location of the brane, which was previously given as $z_5 = 0$, now is at $\sigma v_1 v_2 v_3 v_4 = 0$ – or rather, in our notation, at $z_5 v_1 v_2 v_3 v_4 = 0$.

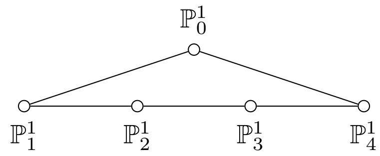


Figure 6.1: Extended Dynkin diagram $\tilde{A}_4$. A line between two points in the diagram means that the respective fibers intersect in exactly one point.

First, let us have a look at what exactly is meant by for example $\mathbb{P}^1_1$. In the 6-dimensional $\tilde{T}_6$, the Weierstrass equation and the equation defining the base, see (6.16), together with $v_1 = 0$ define a 3-fold which we claim is $\mathbb{P}^1_1$ fibered over $\tilde{S}$. Consequently, $\mathbb{P}^1_1$ is given by those three equations together with two equations eq_a and eq_b defining generic divisors corresponding to the base coordinates. (This means that these divisors are neither the GUT divisor nor one of the enhancement loci, and their intersection number inside the GUT surface is 1.) Writing all this down, using $v_1 = 0$ to simplify (6.16), gives

$$\mathbb{P}^1_1 : \quad \begin{cases} v_1 = 0 \\ \ell y \left(v_3 v_4 y - \mathfrak{b}_5\, xz\right) = 0 \\ Q_{(5,2)} = \text{eq}_a = \text{eq}_b = 0 \end{cases} \tag{6.20}$$

(where $Q_{(5,2)}$ still describes the base, see (6.4) and (6.6)). Crucially for us, this does not split for generic $\mathfrak{b}_5$: $\ell = 0$ or $y = 0$ are not solutions to the Weierstrass equation, as ℓv_1 and $y v_1$ are both in the SR ideal – the Weierstrass equation can be rewritten to the irreducible $v_3 v_4 y = \mathfrak{b}_5\, xz$. We will prove later that this object is, in fact, a $\mathbb{P}^1$.

Before we come to that, let us for example check that $\mathbb{P}^1_2$ and $\mathbb{P}^1_3$ intersect in exactly one point. The intersection is given by the following set of equations in $\tilde{T}_6$:

$$\mathbb{P}^1_2 \cap \mathbb{P}^1_3 : \quad \begin{cases} v_2 = v_3 = 0 \\ yz \left(\mathfrak{b}_5\, \ell x + \mathfrak{b}_3\, z_5^2\, v_1 v_4 z^2\right) = 0 \\ Q_{(5,2)} = \text{eq}_a = \text{eq}_b = 0 \end{cases}$$

This describes in fact only one point: If $v_2 = 0 = v_3$, neither y nor z can be zero.

Next, we need to check those intersections that should be empty. In some cases, this is easy: Intersections $\mathbb{P}^1_1 \cap \mathbb{P}^1_3$, $\mathbb{P}^1_0 \cap \mathbb{P}^1_2$ and $\mathbb{P}^1_0 \cap \mathbb{P}^1_3$ are directly excluded from the SR ideal. More scrutiny is needed for example for the intersection $\mathbb{P}^1_2 \cap \mathbb{P}^1_4$, it is given by

$$\mathbb{P}^1_2 \cap \mathbb{P}^1_4 : \quad \begin{cases} v_2 = v_4 = 0 \\ \mathfrak{b}_5\, \ell xyz = 0 \\ Q_{(5,2)} = \text{eq}_a = \text{eq}_b = 0 \end{cases}$$

This intersection is empty as well, as all of ℓ, x, y and z are not allowed to be zero.

Finally, we come back to the proof that the fiber of e.g. the fibration $\mathbb{P}^1_1 \to \tilde{S}$ is, in fact, a $\mathbb{P}^1$. After setting v_1 to zero, we see from the SR ideal that the coordinates y, z, v_3 and ℓ all can not be zero. We can use the scaling relations to set those coordinates to one. In the case of z, v_3 and ℓ this is easy: We just delete, for every coordinate, the only relation containing that coordinate. In the case of y, we'll first subtract the second-to-last line of (6.17) from all the others in such a way that all other lines don't contain y any more, after that we delete that line.

This leaves us with the equations

$$\begin{cases} Q_{(5,2)} = 0 \\ v_4 - \mathfrak{b}_5 x = 0 \end{cases}$$

in the variety

z_1	z_2	z_3	z_4	z_6	z_5	x	v_2	v_4
1	1	1	2	1	0	2	0	3
0	0	0	1	1	1	2	0	3
0	0	0	0	0	1	1	0	1
0	0	0	0	0	0	1	−1	1

with SR ideal $\langle 123, 46, x\bar{4}, 5\bar{2}\rangle$. Now we note that x can not be zero because then v_4 would have to be zero as well. We set x to one using the procedure described above, afterwards v_4 is fixed by $v_4 = \mathfrak{b}_5$ and can be removed from the list of coordinates as well.

We are left with only the equation $Q_{(5,2)} = 0$ in the variety

z_1	z_2	z_3	z_4	z_6	z_5	v_2
1	1	1	2	1	0	2
0	0	0	1	1	1	2
0	0	0	0	0	1	1

with SR ideal $\langle 123, 46, 5\bar{2}\rangle$. This is a $\mathbb{P}^1$ spanned by the homogeneous coordinates z_5 and v_2 fibered over T_3, together with the defining equation of S in T_3.

Conclusion. Let $\mathbb{P}^1_i$ be defined as above and let $E_i = \{v_i = 0\}$ as a divisor of $\tilde{Y}_4$ like in (4.29) (with $E_0 = \{z_5 = 0\}$). Note that

$$[\mathbb{P}^1_i] = [E_i] \wedge [\mathrm{eq}_a] \wedge [\mathrm{eq}_b] \tag{6.21}$$

(with Poincaré duals taken in $\tilde{Y}_4$). We have shown that the intersection numbers give the Cartan matrix,

$$\mathbb{P}^1_i \cdot E_j = \int_{\tilde{Y}_4} [\mathbb{P}^1_i] \wedge [E_j] = \tilde{C}_{ij} \ . \tag{6.22}$$

Note that

$$\int_{\tilde{Y}_4} [\mathbb{P}^1_i] \wedge [E_j] = \int_{\tilde{Y}_4} [E_i] \wedge [E_j] \wedge [\mathrm{eq}_a] \wedge [\mathrm{eq}_b] \ ,$$

and that $\int_S [\mathrm{eq}_a] \wedge [\mathrm{eq}_b] = 1$ by definition. In general, we get

$$\int_{\tilde{Y}_4} [E_i] \wedge [E_j] \wedge \pi^* \left([\mathcal{B}_a] \wedge [\mathcal{B}_b]\right) = \tilde{C}_{ij} \int_{\iota_* S} [\mathcal{B}_a] \wedge [\mathcal{B}_b] \tag{6.23}$$

for arbitrary divisors of the base $\mathcal{B}_i$, as described in (4.29).

Intersection Products of Exceptional Classes. Equation (6.23) can also be proven more formally by considering the products of the exceptional divisor classes in general: Such an intersection can be expressed as

$$[E_i] \wedge [E_j] = \tilde{C}_{ij} \left([z] + \bar{\mathcal{K}}\right) \wedge \mathcal{S} + w^m_{ij} \, [E_m] \wedge \mathcal{S} + k^m_{ij} \, [E_m] \wedge \bar{\mathcal{K}} + b_{ij} \, [E_2] \wedge [E_4] \ . \tag{6.24}$$

$\mathcal{S} = i^*[z_5] + \sum_{i=1}^4 [E_i]$ is the class of S in B_3 (in the resolved space, remember from (6.19) that S is the locus of $z_5 v_1 \cdots v_4 = 0$). Also, $\bar{\mathcal{K}}$ is the anticanonical class of B_3. Since $\mathfrak{b}_5$ is a section of $\bar{\mathcal{K}}$, $\bar{\mathcal{K}}$ can easily be read off from (6.16):

$$\bar{\mathcal{K}} = [v_3 v_4 \ell y^2] - [\ell x y z] = [z_1] + [z_5] + \sum_{i=1}^4 [E_i] \ .$$

Note that $\mathrm{c}_1(B_3) = [z_1] + [z_5]$ in the space T_4 before resolving the singularities.

The coefficients w^m_{ij}, k^m_{ij} and b_{ij} in (6.24) are tabulated in [9, Tab. 10]. The analysis there is applicable to our example because we can compare our SR ideal (6.18) to [9, Tab. 6] and see that our case is called triangulation T_1 there. Note that (6.18) is the SR ideal of $\tilde{T}_6$, the one of $\tilde{Y}_4$ contains in addition generators like z_5x, z_5y or $v_2\ell$ (because the Weierstrass equation does not have a solution with $z_5 = x = 0$, $z_5 = y = 0$ or $v_2 = \ell = 0$).

The right hand side of (6.23) comes from the $([z] \wedge \mathcal{S})$-term in (6.24). We can understand the derivation in detail using the following basic intersection properties (see [29, App. B.1]):

$$\int_{\tilde{Y}_4} [z] \wedge \pi^* \left([\mathcal{B}_a] \wedge [\mathcal{B}_b] \wedge [\mathcal{B}_c]\right) = \int_{B_3} [\mathcal{B}_a] \wedge [\mathcal{B}_b] \wedge [\mathcal{B}_c] \tag{6.25}$$

$$\int_{\tilde{Y}_4} [E_i] \wedge \pi^* \left([\mathcal{B}_a] \wedge [\mathcal{B}_b] \wedge [\mathcal{B}_c]\right) = 0 \qquad (i \in \{1, \ldots, 4\}) \,. \tag{6.26}$$

The first of these equations comes from the fact that, in every fiber of the fibration, $z = 0$ together with the Weierstraß equation (4.19) has exactly one solution. The second one is obvious because $[E_i]$ has one leg in the fiber and one in the base.

6.4 Hypercharge Flux and Chiral Matter

Hypercharge Flux. We already understood that the curve (6.8)

$$\mathcal{C}_Y = \mathcal{C}_{24} - \mathcal{C}_{13} - \mathcal{C}_{63}$$

is trivial on the base B_3 but non-trivial on the brane S.

Using section 5.3, we can now write down the G_4 flux in F-theory: Fibering the exceptional $\mathbb{P}^1$s over the three curves gives the 4-cycles

$$\begin{aligned} \Theta^i_{13}: &\quad z_1 = 0, \quad z_3 = 0, \quad v_i = 0, \quad \mathrm{eq}_W = 0 \\ \Theta^i_{24}: &\quad z_2 = 0, \quad z_4 = 0, \quad v_i = 0, \quad \mathrm{eq}_W = 0 \\ \Theta^i_{63}: &\quad z_6 = 0, \quad z_3 = 0, \quad v_i = 0, \quad \mathrm{eq}_W = 0 \end{aligned} \tag{6.27}$$

in $\tilde{T}_6$. (eq_W denotes the Weierstrass equation, i.e. the second line of (6.16).) Then, as we know from (5.39),

$$\Theta^Y_C = -2\,\Theta^1_C - 4\,\Theta^2_C - 6\,\Theta^3_C - 3\,\Theta^4_C \tag{6.28}$$

and

$$G^Y_4 = \left[\Theta^Y_{24} - \Theta^Y_{13} - \Theta^Y_{63}\right]_{\tilde{Y}_4} \,. \tag{6.29}$$

Matter Curves and Matter Surfaces. We know from section 4.4 that matter lives on matter curves on the brane where the A_4 singularity is enhanced to A_5 or D_5. According to (4.39), the matter curve **10** is

$$\mathcal{C}_{\mathbf{10}}: \quad z_5 = \mathfrak{b}_5 = 0 \tag{6.30}$$

in B_3, for example.

Since P in (6.11) factorizes, the **5** matter curve splits in two components:

$$\mathcal{C}_{\mathbf{5}M}: \quad z_5 = \mathfrak{b}_3 = 0 \quad \text{and} \tag{6.31}$$

$$\mathcal{C}_{\mathbf{5}H}: \quad z_5 = \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5 = 0 \,. \tag{6.32}$$

Let us have a closer look at how the fiber structure changes over those curves:

- Over the $C_{\mathbf{5}M}$ curve, the exceptional $\mathbb{P}^1_2$ splits: Consider the Weierstrass equation (6.16) for $v_2 = 0 = \mathfrak{b}_3$. It reads
$$\ell y \, (v_3 v_4 y - \mathfrak{b}_5\, xz) = 0 \,. \tag{6.33}$$
y can not be zero because yv_2 is in the SR ideal, but ℓ can be set to zero. Thus $\mathbb{P}^1_2 = \mathbb{P}^1_{2\ell} + \mathbb{P}^1_{2E}$ where $\mathbb{P}^1_{2\ell}$ is the component with $\ell = 0$ and $\mathbb{P}^1_{2E}$ the other one. Examining the intersection structure gives the $\tilde{A}_5$ Dynkin diagram shown in figure 6.2 [29].[3]

[3] Comparing with [29, App. A], we see that our resolved space corresponds to one of the triangulations which are called T_{2j} there. It is actually the triangulation T_{21} as we will learn from figure 6.4.

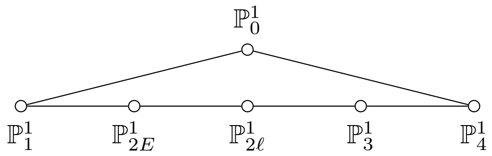


Figure 6.2: Extended Dynkin diagram $\tilde{A}_5$ over $\mathcal{C}_{\mathbf{5}M}$.

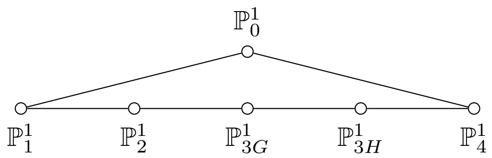


Figure 6.3: Extended Dynkin diagram $\tilde{A}_5$ over $\mathcal{C}_{\mathbf{5}H}$.

Fibered over the matter curve $\mathcal{C}_{\mathbf{5}M}$ this gives the *matter surfaces* $\Sigma^{2\ell}_{\mathbf{5}M}$ and $\Sigma^{2E}_{\mathbf{5}M}$,

$$\Sigma^{2\ell}_{\mathbf{5}M}: \begin{cases} v_2 = \mathfrak{b}_3 = 0 \\ \ell = 0 \\ Q_{(5,2)} = 0 \end{cases} \quad \text{and} \quad \Sigma^{2E}_{\mathbf{5}M}: \begin{cases} v_2 = \mathfrak{b}_3 = 0 \\ v_3 v_4 y - \mathfrak{b}_5\, xz = 0 \\ Q_{(5,2)} = 0 \end{cases} \tag{6.34}$$

in $\tilde{T}_6$.

- Over the $C_{\mathbf{5}H}$ curve, the exceptional $\mathbb{P}^1_3$ splits. To see this, take again the Weierstrass equation (6.16), set $v_3 = 0$ and multiply both sides with $\mathfrak{b}_5$. After using $\mathfrak{b}_2\mathfrak{b}_5 = \mathfrak{b}_3\mathfrak{b}_4$, the result can be written as
$$\left(\mathfrak{b}_5\, \ell xz + \mathfrak{b}_3\, z_5^2\, v_1 v_4 z^3\right) \cdot \left(\mathfrak{b}_5\, y + \mathfrak{b}_4\, z_5\, v_1 v_2 xz\right) = 0\ . \tag{6.35}$$
We will call the two components $\mathbb{P}^1_3 = \mathbb{P}^1_{3G} + \mathbb{P}^1_{3H}$, the corresponding matter surfaces are

$$\Sigma^{3G}_{\mathbf{5}H}: \begin{cases} v_3 = \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5 = 0 \\ \mathfrak{b}_5\, \ell xz + \mathfrak{b}_3\, z_5^2\, v_1 v_4 z^3 = 0 \\ Q_{(5,2)} = 0 \end{cases} \quad \text{and} \quad \Sigma^{3H}_{\mathbf{5}H}: \begin{cases} v_3 = \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5 = 0 \\ \mathfrak{b}_5\, y + \mathfrak{b}_4\, z_5\, v_1 v_2 xz = 0 \\ Q_{(5,2)} = 0 \end{cases}. \tag{6.36}$$

The intersection structure is summarized by the $\tilde{A}_5$ Dynkin diagram in figure 6.3 [29].

Fibering all the other exceptional $\mathbb{P}^1$s over the matter curves gives more matter surfaces, for example

$$\Sigma^{I}_{\mathbf{5}M} = E_I \cap \{\mathfrak{b}_3 = 0\} \quad \text{for} \quad I \in \{0, 1, 3, 4\}\ , \tag{6.37}$$

and similar for $\Sigma^*_{\mathbf{5}H}$. Next, we need to know which linear combinations of the matter surfaces correspond to which states in the $\mathbf{5}_M$ and $\mathbf{5}_H$ representations. This can be read off from the tables in [29, App. A.2]. The vectors of matter surfaces corresponding to the vectors in the $\mathbf{5}$ representations are:

$$\mathbf{\Sigma}_{\mathbf{5}M} = \begin{pmatrix} \Sigma^{2\ell}_{\mathbf{5}M} + \Sigma^{3}_{\mathbf{5}M} + \Sigma^{4}_{\mathbf{5}M} + \Sigma^{0}_{\mathbf{5}M} \\ \Sigma^{2\ell}_{\mathbf{5}M} + \Sigma^{3}_{\mathbf{5}M} + \Sigma^{4}_{\mathbf{5}M} + \Sigma^{0}_{\mathbf{5}M} + \Sigma^{1}_{\mathbf{5}M} \\ \Sigma^{2\ell}_{\mathbf{5}M} \\ \Sigma^{2\ell}_{\mathbf{5}M} + \Sigma^{3}_{\mathbf{5}M} \\ \Sigma^{2\ell}_{\mathbf{5}M} + \Sigma^{3}_{\mathbf{5}M} + \Sigma^{4}_{\mathbf{5}M} \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma}_{\mathbf{5}H} = \begin{pmatrix} \Sigma^{0}_{\mathbf{5}H} + \Sigma^{3H}_{\mathbf{5}H} + \Sigma^{4}_{\mathbf{5}H} \\ \Sigma^{0}_{\mathbf{5}H} + \Sigma^{1}_{\mathbf{5}H} + \Sigma^{3H}_{\mathbf{5}H} + \Sigma^{4}_{\mathbf{5}H} \\ \Sigma^{0}_{\mathbf{5}H} + \Sigma^{1}_{\mathbf{5}H} + \Sigma^{2}_{\mathbf{5}H} + \Sigma^{3H}_{\mathbf{5}H} \\ \Sigma^{3H}_{\mathbf{5}H} + \Sigma^{4}_{\mathbf{5}H} \\ \Sigma^{3H}_{\mathbf{5}H} \end{pmatrix}. \tag{6.38}$$

For later use, let us also mention how the fiber structure over the $\mathcal{C}_{\mathbf{10}M}$ curve changes:

- $\mathbb{P}^1_0$ and $\mathbb{P}^1_3$ don't split.

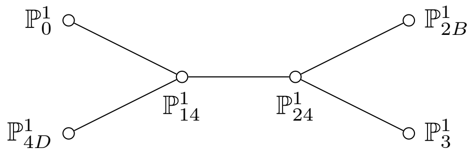


Figure 6.4: Extended Dynkin diagram $\tilde{D}_5$ over $\mathcal{C}_{\mathbf{10}M}$.

- $v_1 = 0$ together with the Weierstrass equation implies that $v_4 = 0$, we say that $\mathbb{P}^1_1 = \mathbb{P}^1_{14}$.
- $\mathbb{P}^1_2 = \mathbb{P}^1_{24} + \mathbb{P}^1_{2B}$ splits into a component with $v_4 = 0$ and one other component.
- $\mathbb{P}^1_4 = \mathbb{P}^1_{14} + \mathbb{P}^1_{24} + \mathbb{P}^1_{4D}$ splits in three components.

The Dynkin diagram is displayed in figure 6.4.

Chiral Indices and Calculating Hypercharges. The *chiral index* $\chi(\mathbf{R}_k)$ of a given state in the representation $\mathbf{R}$ is the number $\nu_+ - \nu_-$ of chiral zero modes. In F-theory it can be calculated by integrating G_4-flux over the corresponding matter surface [5, 29]:

$$\chi(\mathbf{R}_k) = (\mathbf{\Sigma_R})_k \cdot G_4 \ . \tag{6.39}$$

In our case, $G_4 = G_4^Y = F^Y \wedge \omega_Y = [\mathcal{C}_Y]_S \wedge \omega_Y$, where

$$\omega_Y = -2\,[E_1] - 4\,[E_2] - 6\,[E_3] - 3\,[E_4]$$

as discussed in section 5.3. By (6.23), the chiral index is the product of the intersection number of the fibers, and the intersection number in the base:

$$\chi_Y(\mathbf{R}_k) = q^Y(\mathbf{R}_k) \cdot (\mathcal{C}_Y \cdot \mathcal{C}_\mathbf{R}) \ . \tag{6.40}$$

The charge $q^Y(\mathbf{R}_k)$ is the intersection of the fibers. Remembering that $\mathbb{P}^1_3 = \mathbb{P}^1_{3G} + \mathbb{P}^1_{3H}$ and looking at figure 6.3, we get for our example

$$q^Y(\mathbf{5}_H)_5 = \mathbb{P}^1_{3H} \cdot \omega_Y = -2 \cdot 0 - 4 \cdot 0 - 6 \cdot (1-2) - 3 \cdot 1 = 3 \ . \tag{6.41}$$

Similar calculations show that the intersections $\mathbb{P}^1_i \cdot \omega_Y$ vanish for $i \in \{1, 2, 4\}$ and the intersection with $\mathbb{P}^1_0$ is

$$\mathbb{P}^1_0 \cdot \omega_Y = -2 \cdot 1 - 4 \cdot 0 - 6 \cdot 0 - 3 \cdot 1 = -5 \ ,$$

such that

$$q^Y(\mathbf{5}_H) = (-2, -2, -2, 3, 3)^T \tag{6.42}$$

as expected.

The hypercharges of the $\mathbf{5}_M$ states can be calculated in the same way: As

$$(\mathbb{P}^1_0, \mathbb{P}^1_1, \mathbb{P}^1_{2\ell}, \mathbb{P}^1_3, \mathbb{P}^1_4) \cdot \omega_Y = (-5, 0, -2, 5, 0) \ ,$$

we readily get

$$q^Y(\mathbf{5}_M) = (-2, -2, -2, 3, 3)^T \ . \tag{6.43}$$

Splitting the $\mathcal{C}_{\mathbf{5}H}$ Matter Curve. In subsection 3.2.3, we discussed that doublet-triplet splitting is one of the problems of SU(5) GUT theories. The Higgs, transforming in the $\mathbf{5}_H$ of SU(5), transforms as $(\mathbf{1}, \mathbf{2})_{1/2} \oplus (\mathbf{3}, \mathbf{1})_{-1/3}$ under the Standard Model gauge group. The triplet is unwanted because it could for example mediate proton decay. In general, there is no explanation for why the triplet should be absent or much more massive than the doublet. We will see that we can easily achieve doublet-triplet splitting in our model, though, if the respective components have different chiral indices.

So far, the chiral index $\chi^Y(\mathbf{5}_H)$ of the Higgs is zero though, for a simple reason: The curve $\mathcal{C}_{\mathbf{5}H}$ is defined as the total intersection of B_3 with $z_5 = \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5 = 0$ in T_4, hence

$$[\mathcal{C}_{\mathbf{5}H}]_S = [\mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5]_S = \iota^*[z_5 \cap \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5]_{B_3} = \iota^*\omega \tag{6.44}$$

for $\omega \in H^{1,1}(B_3)$. The hypercharge flux was constructed in such a way that $\int_{\mathcal{C}_Y} \iota^*\omega = 0$, therefore

$$(\boldsymbol{\Sigma}_{\mathbf{5}H})_k \cdot G_4^Y = 0 \; . \tag{6.45}$$

If on the other hand we split the curve $\mathcal{C}_{\mathbf{5}H}$, we can achieve different chiral indices for the doublet and the triplet. Remember that $\mathcal{C}_{\mathbf{5}H}$ is given by

$$\mathcal{C}_{\mathbf{5}H}: \quad z_5 = 0\,, \quad \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5 = 0 \quad \text{and} \quad Q_{(5,2)} = z_1z_2z_6\,F_1 + z_3z_4\,\tilde{F}_1 = 0$$

in T_4, the last equation being (6.6). If now

$$\mathfrak{b}_3 = \hat{\mathfrak{b}}_3\,z_2 + \mathfrak{b}_5\,Q_2(z_1,z_2,z_3) \quad \text{and} \quad \mathfrak{b}_2 = \hat{\mathfrak{b}}_2\,z_4 + \mathfrak{b}_4\,Q_2(z_1,z_2,z_3)\,, \tag{6.46}$$

$$\text{then } \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5 = \hat{\mathfrak{b}}_3\mathfrak{b}_4\,z_2 - \hat{\mathfrak{b}}_2\mathfrak{b}_5\,z_4 \tag{6.47}$$

such that $z_2 = z_4 = 0$ automatically satisfies both given conditions. Therefore, $\mathcal{C}_{\mathbf{5}H}$ splits into two components $\mathcal{C}_{\mathbf{5}H_u}$ and $\mathcal{C}_{\mathbf{5}H_d}$, where

$$\mathcal{C}_{\mathbf{5}H_d} = \{z_2 = z_4 = z_5 = 0\} \quad \text{and} \quad \mathcal{C}_{\mathbf{5}H_u} = \mathcal{C}_{\mathbf{5}H} - \mathcal{C}_{\mathbf{5}H_d}\,. \tag{6.48}$$

The matter surfaces split accordingly, $\boldsymbol{\Sigma}_{\mathbf{5}H} = \boldsymbol{\Sigma}_{\mathbf{5}H_u} + \boldsymbol{\Sigma}_{\mathbf{5}H_d}$, and the integration yields

$$\chi_Y(\mathbf{5}_{H_d}) = q^Y(\mathbf{5}_H)\cdot(\mathcal{C}_Y\cdot\mathcal{C}_{\mathbf{5}H_d}) = \begin{pmatrix} 2\\2\\2\\-3\\-3 \end{pmatrix} \quad \text{and} \quad \chi_Y(\mathbf{5}_{H_u}) = \chi_Y(\mathbf{5}_H) - \chi_Y(\mathbf{5}_{H_d}) = \begin{pmatrix} -2\\-2\\-2\\3\\3 \end{pmatrix}\,. \tag{6.49}$$

We used that $\mathcal{C}_{\mathbf{5}H_d} = \mathcal{C}_{24}$, and we could simply use the already known intersection numbers:

$$(\mathcal{C}_{24} - \mathcal{C}_{13} - \mathcal{C}_{63})\cdot\mathcal{C}_{24} = \mathcal{C}_{24}^2 = -1\,.$$

Note that the integral is allowed to be non-zero here, because $\mathcal{C}_{\mathbf{5}H_d}$ can not be written as a complete intersection involving B_3.

6.5 Doublet-Triplet Splitting

$\mathrm{U}(1)_X$-Flux and Charges. So far we have neglected the additional $\mathrm{U}(1)_X$ gauge group. We will now switch on a $\mathrm{U}(1)_X$ flux, that means taking

$$G_4 = G_4^X + G_4^Y \quad \text{for} \quad G_4^X = F_X \wedge \omega_X\,, \tag{6.50}$$

where F_X is a two-form in the base like in (5.36). Let $L = \{\ell = 0\}$ and $Z = \{z = 0\}$, then the two-form ω_X is given by [1,9,29]

$$\omega_X = -5\,([L] - [Z] - \bar{\mathcal{K}}) - 2\,[E_1] - 4\,[E_2] - 6\,[E_3] - 3\,[E_4] = \omega_Y - 5\,[E_5]\,, \tag{6.51}$$

where $[E_5] = [L] - [Z] - \bar{\mathcal{K}}$. We choose

$$F_X = i^*\,([z_1] - 8\,[z_6]) \tag{6.52}$$

(i being the embedding $i : B_3 \to T_4$).

Before going on, we'd like to calculate the $\mathrm{U}(1)_X$ charges of the different matter surfaces. If we want to be able to do that, we first need to know how the divisors L and Z intersect the various $\mathbb{P}^1_*$ fibers.

- Over the $\mathcal{C}_{\mathbf{5}M}$ curve, the Weierstrass equation reads

$$0 = \mathfrak{b}_2\, z_5^3\, v_1^2 v_2 v_4\, x z^4 \ .$$

Taking the SR ideal into consideration, this means that the fiber $\mathbb{P}^1_\ell$ splits into the two components, $\mathbb{P}^1_\ell = \mathbb{P}^1_{2\ell} + \mathbb{P}^1_{\ell x}$, where $\mathbb{P}^1_{2\ell}$ is the object we already encountered before. This lets us calculate

$$(\mathbb{P}^1_0, \mathbb{P}^1_1, \mathbb{P}^1_{2\ell}, \mathbb{P}^1_3, \mathbb{P}^1_4) \cdot L = (0, 0, -2+1, 1+0, 0) = (0, 0, -1, 1, 0) \ .$$

Further, the divisor Z intersects none of the $\mathbb{P}^1_i$ for $i \neq 0$ and it turns out that $\mathbb{P}^1_0 \cdot Z = 1$. Because $\bar{\mathcal{K}}$ is a class of the base, its intersections with the exceptional $\mathbb{P}^1$s are zero (remember (6.26)). It follows that

$$(\mathbb{P}^1_0, \mathbb{P}^1_1, \mathbb{P}^1_{2\ell}, \mathbb{P}^1_3, \mathbb{P}^1_4) \cdot E_5 = (-1, 0, -1, 1, 0)$$

and ultimately $(\mathbb{P}^1_0, \mathbb{P}^1_1, \mathbb{P}^1_{2\ell}, \mathbb{P}^1_3, \mathbb{P}^1_4) \cdot \omega_X = (0, 0, 3, 0, 0)$ such that

$$q^X(\mathbf{5}_M) = +3 \quad \text{for all components.} \tag{6.53}$$

- Over the $\mathcal{C}_{\mathbf{5}H}$ curve, the intersections simply turn out to be:

$$(\mathbb{P}^1_0, \mathbb{P}^1_1, \mathbb{P}^1_2, \mathbb{P}^1_{3H}, \mathbb{P}^1_4) \cdot L = (0, 0, 0, 1, 0)$$
$$(\mathbb{P}^1_0, \mathbb{P}^1_1, \mathbb{P}^1_2, \mathbb{P}^1_{3H}, \mathbb{P}^1_4) \cdot Z = (1, 0, 0, 0, 0) \ .$$

Hence, $(\mathbb{P}^1_0, \mathbb{P}^1_1, \mathbb{P}^1_2, \mathbb{P}^1_{3H}, \mathbb{P}^1_4) \cdot \omega_X = (0, 0, 0, -2, 0)$ and

$$q^X(\mathbf{5}_H) = -2 \ . \tag{6.54}$$

- For the $\mathcal{C}_{\mathbf{10}H}$ curve, let's for example consider the component of the matter surface with fiber $\mathbb{P}^1_{4D}$. Its hypercharge is $q^Y = -2 - 0 - 0 - 3(-2+1) = +1$, and $q^X = +1$ as well since L and Z do not intersect $\mathbb{P}^1_{4D}$. Analysis along these lines shows that, again, all components have the same $\mathrm{U}(1)_X$ charge,

$$q^X(\mathbf{10}_M) = +1 \ . \tag{6.55}$$

Doublet-Triplet-Splitting. We are finally able to implement doublet-triplet splitting. Using again the formula $\chi_X(\mathbf{R}_k) = q^X(\mathbf{R}_k) \cdot \int_S F_X \wedge [\mathcal{C}_\mathbf{R}]_S$, we calculate[4] the chiral indices of the Higgs vectors:

$$\begin{aligned}
\chi_X(\mathbf{5}_{H_d}) &= -2 \int_{\mathcal{C}_{\mathbf{5}H_d}} F_X = -2 \int_{\mathcal{C}_{\mathbf{5}H_d}} i^*\left([z_1] - 8\,[z_6]\right) = -2 \int_{i_*\mathcal{C}_{\mathbf{5}H_d}} \left([z_1] - 8\,[z_6]\right) \\
&= -2 \int_{T_4} [z_2] \wedge [z_4] \wedge [z_5] \wedge \left([z_1] - 8\,[z_6]\right) = -2 \ , \qquad (6.56)\\
\chi_X(\mathbf{5}_{H_u}) &= -2 \int_{T_4} \big([Q_{(5,2)}] \wedge [\mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5] - [z_2] \wedge [z_4]\big) \wedge [z_5] \wedge (8[z_6] - [z_1]) = +2 \ . \qquad (6.57)
\end{aligned}$$

Therefore, finally,

$$\chi(\mathbf{5}_{H_d}) = \chi_X(\mathbf{5}_{H_d}) + \chi_Y(\mathbf{5}_{H_d}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -5 \\ -5 \end{pmatrix} \quad \text{and} \quad \chi(\mathbf{5}_{H_u}) = \chi_X(\mathbf{5}_{H_u}) + \chi_Y(\mathbf{5}_{H_u}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 5 \end{pmatrix} \ . \tag{6.58}$$

There are five Higgs doublets H_d in the $(\bar{\mathbf{1}}, \bar{\mathbf{2}})_{-1/2} = (\mathbf{1}, \mathbf{2})_{-1/2}$ of the Standard Model and five Higgs doublets H_u in the $(\mathbf{1}, \mathbf{2})_{1/2}$, this is five times the spectrum of the MSSM (compare subsection 3.1.2). As required, there are no Higgs triplets.

[4] All such integrals were evaluated using Sage, see subsection D.3.1.

Number of generations. Let us count the generations of matter on the $\mathbf{5}_M$ and $\mathbf{10}_M$ curves. Like the Higgs surface, the matter surfaces $\mathbf{\Sigma_{5}}_M$ and $\mathbf{\Sigma_{10}}_M$ don't intersect the hypercharge flux G_4^Y because the matter curves can be written as a complete intersection involving B_3. Therefore $\chi_Y(\mathbf{5}_M) = \chi_Y(\mathbf{10}_M) = 0$ and we only need to calculate $\chi_X(\mathbf{5}_M)$ and $\chi_X(\mathbf{10}_M)$.

Using the charges calculated above, we get

$$\chi(\mathbf{5}_M) = \chi_X(\mathbf{5}_M) = 3\int_{T_4} [Q_{(5,2)}] \wedge [\mathfrak{b}_3] \wedge [z_5] \wedge ([z_1] - 8\,[z_6]) = -9\ , \tag{6.59}$$

$$\chi(\mathbf{10}_M) = \chi_X(\mathbf{10}_M) = \int_{T_4} [Q_{(5,2)}] \wedge [\mathfrak{b}_5] \wedge [z_5] \wedge ([z_1] - 8\,[z_6]) = 9\ . \tag{6.60}$$

Hence, we have 9 generations of matter, each contained in a $\bar{\mathbf{5}} \oplus \mathbf{10}$ as discussed around (3.9).

Finally, there is some more chiral matter on the matter surface over the conifold singularity,

$$\Sigma_{\mathbf{1}} = \{Q_{(5,2)} = \mathfrak{b}_2 = \mathfrak{b}_3 = \ell = 0\}\ . \tag{6.61}$$

The number of singlets is [1]

$$\chi(\mathbf{1}) = 5\int [Q_{(5,2)}] \wedge [\mathfrak{b}_2] \wedge [\mathfrak{b}_3] \wedge ([z_1] - 8\,[z_6]) = -1095\ . \tag{6.62}$$

6.6 Improving the Model

Using only the methods introduced so far, the result obtained above is optimal: We could install a free parameter η in the G_4^Y-flux, modifying (6.29) to

$$G_4^Y = \eta\,\left[\Theta_{24}^Y - \Theta_{13}^Y - \Theta_{63}^Y\right]_{\tilde{Y}_4}\ . \tag{6.63}$$

Further, we could generalize the definition of G_4^X (6.52) to

$$F_X = i^*\,(a\,[z_1] + b\,[z_6]) \tag{6.64}$$

with free parameters a and b, this is the most general form of a class in B_3.

For the reasons discussed above,

$$\chi_Y(\mathbf{5}_M) = \chi_Y(\mathbf{10}_M) = \chi_Y(\mathbf{1}) = 0\ ,$$

and obviously

$$\chi_Y(\mathbf{5}_{H_d}) = \eta\,(2,2,2,-3,-3)^T\ , \quad \chi_Y(\mathbf{5}_{H_u}) = -\chi_Y(\mathbf{5}_{H_d})\ .$$

Generalizing the calculations of χ_X above yields

$$\begin{aligned}
\chi_X(\mathbf{5}_{H_d}) &= -2a\\
\chi_X(\mathbf{5}_{H_u}) &= -14a - 2b\\
\chi_X(\mathbf{5}_M) &= 15a + 3b\\
\chi_X(\mathbf{10}_M) &= a - b\\
\chi_X(\mathbf{1}) &= 65a + 145b\ .
\end{aligned} \tag{6.65}$$

Doublet-triplet splitting requires $2\eta - 2a = 0$ and $-2\eta - 14a - 2b = 0$, we have to take $a = \eta$ and $b = -8\eta$. Then

$$\chi(\mathbf{5}_{H_{u/d}}) = (0,0,0,\pm 5\eta, \pm 5\eta)^T\ , \tag{6.66}$$

$$\chi(\mathbf{5}_M) = -9\eta\ , \tag{6.67}$$

$$\chi(\mathbf{10}_M) = 9\eta\ , \tag{6.68}$$

$$\chi(\mathbf{1}) = -1095\eta\ . \tag{6.69}$$

Since chiral indices should be integers, the choice $\eta = 1$ used above is the best possibility.

In the following we will show that the result can still be improved.

Introducing G_4^λ-Flux. There is one more G_4-flux in F-theory [9], it has been neglected so far: Let $\omega_\lambda = 2\,[E_1] - [E_2] + [E_3] - 2\,[E_4]$, then

$$G_4^\lambda = \lambda\,\left(5\,[E_2] \wedge [E_4] + \omega_\lambda \wedge \bar{\mathcal{K}}\right)\ , \tag{6.70}$$

where λ is a free parameter. The so-defined object satisfies all constraints that G_4-fluxes have to obey, especially:

- It is orthogonal to Z, i.e. $\int_{\tilde{Y}_4} G_4^\lambda \wedge Z \wedge [\mathcal{B}_a] = 0$ for any base divisor $\mathcal{B}_a$. This follows because the SR ideal contains zv_i for all i.
- It is orthogonal to all base divisors, i.e. $\int_{\tilde{Y}_4} G_4^\lambda \wedge [\mathcal{B}_a] \wedge [\mathcal{B}_b] = 0$. The reason is that we can use (6.23), and $\tilde{C}_{24} = 0$.
- It is orthogonal to all Cartan fluxes $G_4^i = [E_i] \wedge F_i$. This can be proven from the intersection numbers [9]

$$\int_{\tilde{Y}_4} [E_2] \wedge [E_4] \wedge [E_i] \wedge [\mathcal{B}_a] = (1,-1,1,-1)_i \int_S \mathrm{c}_1(B_3) \wedge [\mathcal{B}_a]\ .$$

(They can be derived from the intersection products (6.24).)
Hence we get $\int_{\tilde{Y}_4} (5\,[E_2] \wedge [E_4]) \wedge G_4^i = (5,-5,5,-5)_i \int_S \mathrm{c}_1(B_3) \wedge F_i$, and on the other hand

$$\begin{aligned}\int_{\tilde{Y}_4} \omega_\lambda \wedge \bar{\mathcal{K}} \wedge [E_i] \wedge F_i &= (2C_{1i} - C_{2i} + C_{3i} - 2C_{4i}) \int_S \mathrm{c}_1(B_3) \wedge F_i \\ &= (-5,5,-5,5)_i \int_S \mathrm{c}_1(B_3) \wedge F_i\ ,\end{aligned}$$

such that

$$\int_{\tilde{Y}_4} G_4^\lambda \wedge G_4^i = 0\ . \tag{6.71}$$

The last property notably implies that the chiral indices of all components of matter in a certain representation (like $\mathbf{5}_{H_{u/d}}$, $\mathbf{5}_M$ or $\mathbf{10}_M$) are equal: The difference between components is always a sum of matter surfaces of the form $[E_i] \wedge \cdots$.

Chiral Indices. It remains to calculate the chiral indices $\chi_\lambda(\mathbf{R}) = \mathbf{\Sigma_R} \cdot G_4^\lambda$. This is slightly more difficult than the previous calculations, because G_4^λ is not of the type $F \wedge \omega$. We've seen above how to handle intersection products involving two exceptional divisors $\int_{\tilde{Y}_4} [E_i] \wedge [E_j] \wedge [\mathcal{B}_a] \wedge [\mathcal{B}_b]$ or three exceptional divisors $\int_{\tilde{Y}_4} [E_2] \wedge [E_4] \wedge [E_i] \wedge \mathcal{B}_a]$, both can be easily reduced to integrals over B_3 using the methods displayed in [9, App. B]. We have also been able to deal with products of the type $\int_{\tilde{Y}_4} [E_i] \wedge [\mathcal{B}_a] \wedge [\Sigma^{2\ell}_{\mathbf{5}M}]$ by looking at the intersection structure related to the Dynkin diagram of the enhanced gauge symmetry.

The ingredient which is still missing are integrals like $\int_{\tilde{Y}_4} [E_2] \wedge [E_4] \wedge [\Sigma^{2\ell}_{\mathbf{5}M}]$. In principle, this can be done calculating in $\tilde{T}_6$, using the homological relations between the divisor classes and the relations following from the SR ideal in order to write down the intersection ring of $\tilde{T}_6$. In that way, identities like

$$\begin{aligned}\chi_\lambda(\mathbf{10}_M) &= \lambda \int_{\mathcal{C}_{\mathbf{10}M}} (-6\bar{\mathcal{K}} + 5\mathcal{S}) \\ \chi_\lambda(\mathbf{5}_M) &= 2\lambda \int_{\mathcal{C}_{\mathbf{5}M}} \bar{\mathcal{K}} \\ \chi_\lambda(\mathbf{5}_H) &= -\lambda \int_{B_3} \mathcal{S} \wedge \mathcal{S} \wedge \bar{\mathcal{K}} \\ \chi_\lambda(\mathbf{1}) &= 0\end{aligned} \tag{6.72}$$

listed in [9, Table 1] can be derived.

We will simply use a computer algebra system to automatically calculate the integrals over $\tilde{T}_6$ (see section D.3 in the appendix). The results are as follows:

- For $\mathbf{5}_H$, see (6.36) to get

$$\chi_\lambda(\mathbf{5}_H) = \int_{\tilde{T}_6} [Q_{(5,2)}] \wedge ([\mathfrak{b}_5] + [y]) \wedge [E_3] \wedge ([\mathfrak{b}_3] + [\mathfrak{b}_4]) \wedge G_4^\lambda = 2\lambda \, . \tag{6.73}$$

  Note that we know from the definition that $[\mathfrak{b}_{5-k}] = \bar{\mathcal{K}} + k\,(\bar{\mathcal{K}} - \mathcal{S})$.

- For $\mathbf{5}_{H_d}$, we need to take a closer look at the surface $\Sigma^{3H}_{\mathbf{5}H_d}$. Since is given by the equations $v_3 = 0 = \mathfrak{b}_5\, y + \mathfrak{b}_4\, z_5\, v_1 v_2 x z$ from in (6.36), together with $z_2 = 0 = z_4$ from (6.48), we need to integrate

$$\chi_\lambda(\mathbf{5}_{H_d}) = \int_{\tilde{T}_6} ([\mathfrak{b}_5] + [y]) \wedge [E_3] \wedge [z_2] \wedge [z_4] \wedge G_4^\lambda = 0 \, . \tag{6.74}$$

  This implies $\chi_\lambda(\mathbf{5}_{H_u}) = \chi_\lambda(\mathbf{5}_H) = 2\lambda$.

- For $\mathbf{5}_M$, we remember (6.34) and the result is

$$\chi_\lambda(\mathbf{5}_M) = \int_{\tilde{T}_6} [Q_{(5,2)}] \wedge [\ell] \wedge [E_2] \wedge [\mathfrak{b}_3] \wedge G_4^\lambda = 2\lambda \, . \tag{6.75}$$

- As mentioned above, one of the components in $\mathbf{10}_M$ has $\mathbb{P}^1_{4D}$ as its fiber, the matter surface is given by the equations

$$\Sigma^{4D}_{\mathbf{10}M} : \quad Q_{(5,2)} = (v_2 v_3 \ell x + \mathfrak{b}_4 z_5 z^2) = v_4 = \mathfrak{b}_5 = 0$$

  in $\tilde{T}_6$. Using this, we can determine the respective chiral index

$$\chi_\lambda(\mathbf{10}_M) = \int_{\tilde{T}_6} [Q_{(5,2)}] \wedge ([v_2] + [v_3] + [\ell] + [x]) \wedge [E_4] \wedge [\mathfrak{b}_5] \wedge G_4^\lambda = -4\lambda \, . \tag{6.76}$$

- And finally, $\chi_\lambda(\mathbf{1}) = 0$.

Optimal Spectrum. Including G_4^λ-flux, the chiral indices are[5]

$$\chi(\mathbf{5}_{H_d}) = -2a + (2,-3)\,\eta \tag{6.77}$$
$$\chi(\mathbf{5}_{H_u}) = -14a - 2b + (-2,3)\,\eta + 2\lambda \tag{6.78}$$
$$\chi(\mathbf{5}_M) = 15a + 3b + 2\lambda \tag{6.79}$$
$$\chi(\mathbf{10}_M) = a - b - 4\lambda \tag{6.80}$$
$$\chi(\mathbf{1}) = 65a + 145b \, . \tag{6.81}$$

Doublet-triplet splitting requires that $a = \eta$ and $b = \lambda - 8\eta$. Implementing this, we get

$$\begin{aligned} &\chi(\mathbf{5}_{H_d}) = (0,-5\eta)\,, \quad \chi(\mathbf{5}_{H_u}) = (0,5\eta)\,, \\ &\chi(\mathbf{5}_M) = 5\lambda - 9\eta\,, \quad \chi(\mathbf{10}_M) = -5\lambda + 9\eta\,, \\ &\chi(\mathbf{1}) = 145\lambda - 1095\eta\,. \end{aligned}$$

Requiring that all those numbers are integers, there is no solution with one Higgs doublet, i.e. $5\eta = 1$. Instead, we can require that there are three generations of matter. Using $-5\lambda + 9\eta = 3$, we can eliminate λ and get

$$\begin{aligned} &\chi(\mathbf{5}_{H_d}) = (0,-5\eta)\,, \quad \chi(\mathbf{5}_{H_u}) = (0,5\eta)\,, \\ &\chi(\mathbf{5}_M) = -3\,, \quad \chi(\mathbf{10}_M) = 3\,, \\ &\chi(\mathbf{1}) = -834\eta - 87\,. \end{aligned}$$

Since 834 and 5 don't have common factors, the best we can do is to choose $\eta = 1$ which gives five Higgs doublets.

[5] We simply write e.g. -2 instead of $(-2,-2,-2,-2,-2)^T$, and $(2,-3)$ instead of $(2,2,2,-3,-3)$ for readability.

Splitting the Higgs-Curve Differently. We made a choice above when we defined

$$\mathcal{C}_{H_d} = \mathcal{C}_{24} = \{z_2 = z_4 = z_5 = 0\} \tag{6.82}$$

in (6.48).

If we were to take, for example,

$$\mathfrak{b}_3 = \hat{\mathfrak{b}}_3\, z_1 + \mathfrak{b}_5\, Q_2(z_1, z_2, z_3) \quad \text{and} \quad \mathfrak{b}_2 = \hat{\mathfrak{b}}_2\, z_3 + \mathfrak{b}_4\, Q_2(z_1, z_2, z_3) \tag{6.83}$$

instead of (6.46), then the Higgs curve would split into

$$\mathcal{C}_{H_d} = \mathcal{C}_{13} = \{z_1 = z_3 = z_5 = 0\} \tag{6.84}$$

and $\mathcal{C}_{H_u} = \mathcal{C}_H - \mathcal{C}_{H_d}$. Note that also $\mathcal{C}_{13}$ implicitly solves $Q_{(5,2)} = 0$.

In fact, there are three choices for $\mathcal{C}_{H_d}$: Either $\mathcal{C}_{24}$, as explored above, or $\mathcal{C}_{13}$, or $\mathcal{C}_{63}$. Note that there is no $\mathcal{C}_{64}$ because of the SR ideal.

- We can quickly discard the case $\mathcal{C}_{H_d} = \mathcal{C}_{63}$ because

$$\mathcal{C}_Y \cdot \mathcal{C}_{63} = \mathcal{C}_{24} \cdot \mathcal{C}_{63} - \mathcal{C}_{13} \cdot \mathcal{C}_{63} - \mathcal{C}_{63}^2 = 0 - 1 - (-1) = 0\ , \tag{6.85}$$

  which means that there can not be doublet-triplet splitting.

- In the case $\mathcal{C}_{H_d} = \mathcal{C}_{13}$ we get the chiral indices

$$\chi(\mathbf{5}_{H_d}) = -2b + (-2, 3)\, \eta + 2\lambda \tag{6.86}$$
$$\chi(\mathbf{5}_{H_u}) = -16a + (2, -3)\, \eta \tag{6.87}$$

  and the unchanged (6.79), (6.80), (6.81). Note that the sign in front of the η-terms changed compared to before because $\mathcal{C}_Y \cdot \mathcal{C}_{13} = +1$. The calculations of χ_X and χ_λ can be found in subsections D.3.1 and D.3.2, respectively.

  Proceeding like above, we see that doublet-triplet splitting requires $\eta = 8a$ and $b = \lambda - 8a$, leaving a and λ as free parameters. Again, there is no solution with only one Higgs doublet, but a solution with three generations of chiral matter. With that choice, we get twenty Higgs doublets, however.

Chapter 7

Summary

Hypercharge Flux. We used the toric variety with scaling relations

$$\begin{array}{cccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ \hline 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \tag{7.1}$$

and SR ideal $\langle z_1 z_2 z_3, z_4 z_5 z_6 \rangle$ as the ambient geometry for our calculations. The base manifold B_3 of the elliptic fibration is the zero locus of a polynomial of scaling degree $(5, 2)$ and the brane locus S is the intersection of B_3 with $\{z_5 = 0\}$. Note that $\{z_5 = 0\}$ is the blow-up of a $\mathbb{P}^3$ in a point, therefore S is a dP_7-surface.

The defining equation of B_3 was restricted to have the form

$$z_1 z_2 z_6 \, F_1(z_1 z_6, z_2 z_6, z_3 z_6, z_4) + z_3 z_4 \, \tilde{F}_1(z_1 z_6, z_2 z_6, z_3 z_6, z_4) + \mathcal{O}(z_5) = 0 \tag{7.2}$$

(where F_1 and $\tilde{F}_1$ are linear combinations of their arguments). Then, the curves

$$\begin{aligned} \mathcal{C}_{13} : &\quad \{z_1 = z_3 = z_5 = 0\} \ , \\ \mathcal{C}_{24} : &\quad \{z_2 = z_4 = z_5 = 0\} \text{ and} \\ \mathcal{C}_{63} : &\quad \{z_6 = z_3 = z_5 = 0\} \end{aligned} \tag{7.3}$$

lie in S and the combination

$$\mathcal{C}_Y = \mathcal{C}_{24} - \mathcal{C}_{13} - \mathcal{C}_{63} \tag{7.4}$$

is trivial in B_3 but non-trivial in S. In other words, it satisfies our condition (1.1)

$$\iota_* \mathcal{C}_Y = 0 \ , \tag{7.5}$$

so that a hypercharge flux $F^Y = [\mathcal{C}_Y]_S$ breaks an SU(5) GUT to the Standard Model.

We modeled the SU(5) GUT in F-theory by writing down a U(1)-restricted Tate model with appropriately restricted coefficients, precisely

$$a_1 = \mathfrak{b}_5 \ , \quad a_2 = \mathfrak{b}_4 \, z_5 \ , \quad a_3 = \mathfrak{b}_3 \, z_5^2 \ , \quad a_4 = \mathfrak{b}_2 \, z_5^3 \ , \quad a_6 = 0 \ , \tag{7.6}$$

where the $\mathfrak{b}_i$ are sections which do not have global factors of z_5. This describes a singular elliptic fibration. The resolved variety is a bundle where the fiber over a point in S consists of five $\mathbb{P}^1$s intersecting like the extended Dynkin diagram of SU(5). If Θ^i_Y is the surface which is the i-th such $\mathbb{P}^1$ fibered over the curve $\mathcal{C}_Y$ from above, the hypercharge flux F^Y corresponds to

$$G_4^Y = \left[-2\Theta^1_Y - 4\Theta^2_Y - 6\Theta^3_Y - 3\Theta^4_Y \right]_{\tilde{Y}_4} \tag{7.7}$$

flux in F-theory.

Doublet-Triplet Splitting. In F-theory matter lives on matter surfaces Σ, which are linear combinations of the exceptional $\mathbb{P}^1$s fibered over matter curves. For example, the Higgs matter curve is

$$\mathcal{C}_{\mathbf{5}H}: \quad z_5 = \mathfrak{b}_3\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_5 = 0 \tag{7.8}$$

and the different components of the Higgs vector correspond to different matter surfaces $(\Sigma_{\mathbf{5}H})_k$ over this curve. The chiral index $\chi\left(\mathbf{5}_H\right)_k$, i.e. the number of zero modes, can be counted by integrating G_4-flux over the surface,

$$\chi(\mathbf{R}_k) = \left(\mathbf{\Sigma_R}\right)_k \cdot G_4 \ . \tag{7.9}$$

By definition of G_4^Y, $\left(\mathbf{\Sigma}_{\mathbf{5}H}\right)_k \cdot G_4^Y = 0$. Nevertheless, we were able to achieve different chiral indices for the doublet and the triplet: We split the Higgs matter curve by restricting the form of the parameters $\mathfrak{b}_2$ and $\mathfrak{b}_3$ in the following way:

$$\mathfrak{b}_2 = \hat{\mathfrak{b}}_2\, z_4 + \mathfrak{b}_4\, Q_2(z_1,z_2,z_3) \quad \text{and} \quad \mathfrak{b}_3 = \hat{\mathfrak{b}}_3\, z_2 + \mathfrak{b}_5\, Q_2(z_1,z_2,z_3) \ . \tag{7.10}$$

Then $\mathcal{C}_{\mathbf{5}H}$ splits into $\mathcal{C}_{\mathbf{5}H_d} = \mathcal{C}_{24}$ and $\mathcal{C}_{\mathbf{5}H_u} = \mathcal{C}_{\mathbf{5}H} - \mathcal{C}_{24}$. For each Higgs vector, the doublet and the triplet have different chiral indices:

$$\chi\left(\mathbf{5}_{H_d}\right) = (2,-3) \quad \text{and} \quad \chi\left(\mathbf{5}_{H_u}\right) = (-2,3) \tag{7.11}$$

(the first number stands for the chiral index of the triplet and the second is that of the doublet).

Final Spectrum. In the end, we assumed the most general form of G_4-flux (compatible with the breaking of SU(5) to the Standard Model):

$$G_4 = \eta\, G_4^Y + G_4^X + G_4^\lambda \ . \tag{7.12}$$

The hypercharge flux was scaled with a free parameter η, and G_4^X and G_4^λ are the only G_4-fluxes which are orthogonal to all Cartan fluxes, see (6.64) and (6.70). G_4^X has two free parameters a and b corresponding to the two classes in $H^2(B_3)$, G_4^λ has one free parameter λ.

The chiral indices of all matter representations contained in our model are:

$$\chi(\mathbf{5}_{H_d}) = (2\eta - 2a,\ -3\eta - 2a) \ , \tag{7.13}$$
$$\chi(\mathbf{5}_{H_u}) = (2\lambda - 2\eta - 14a - 2b,\ 2\lambda + 3\eta - 14a - 2b) \ , \tag{7.14}$$
$$\chi(\mathbf{5}_M) = 2\lambda + 15a + 3b \ , \tag{7.15}$$
$$\chi(\mathbf{10}_M) = -4\lambda + a - b \ , \tag{7.16}$$
$$\chi(\mathbf{1}) = 65a + 145b \ . \tag{7.17}$$

(The different components of the $\mathbf{5}_M$ and $\mathbf{10}_M$ representations all have the same chiral index.) All of these numbers must be integer, and for doublet-triplet splitting the first entries in (7.13) and (7.14) must be zero. These constraints lead us to the following conclusions:

- Without G_4^λ-flux (for $\lambda = 0$), the result stated in [1] is optimal: We can at best get down to 9 generations of chiral matter and 5 Higgs doublets.
- By introducing G_4^λ-flux, we are able to reduce the number of generations to the MSSM value of 3. The number of Higgs doublets is still 5, though.
- We also explored all other possible choices for splitting the Higgs matter curve, but found that the other choices are either inconsistent or lead to a higher number of Higgs doublets.

Mathematical Appendix

Appendix A

Differential Geometry

Note. The contents of this chapter were compiled from [53, Ch. I.2 - II.3], [54, Ch. 3 - 7, 9 - 10] and [55].

A.1 Manifolds

A.1.1 Definition

Definition A.1 (Differentiable Manifold)**.** Let M be a topological space and $m \in \mathbb{N}_+$. A pair (U, φ) where $U \subset M$ is open and φ is a homeomorphism from U to $\varphi(U) \subset \mathbb{R}^m$ is called a *chart*. A family $\{(U_i, \varphi_i)\}$ of charts (of identical dimension) such that $\bigcup_i U_i = M$ is called an *atlas* of M.

M is called an m-dimensional *differentiable manifold* if it is equipped with an atlas such that the transition functions $\varphi_i \circ \varphi_j^{-1}$ are smooth (C^∞) where they are defined.

Two atlases are called compatible if all transition functions of their union are smooth where they are defined. An equivalence class of compatible atlases on M is called a *differentiable structure*.

Definition A.2 (Vector Field)**.** Let M be a differentiable manifold. The set $C^\infty(M)$ of *smooth functions* consists of functions $f : M \to \mathbb{R}$ such that $f \circ \varphi_i^{-1}$ is smooth for every chart U_i.

A *vector field* v on M is an $\mathbb{R}$-linear function $v : C^\infty(M) \to C^\infty(M)$ satisfying the Leibniz law

$$v(f \cdot g) = v(f) \cdot g + f \cdot v(g) \ . \tag{A.1}$$

The set of all vector fields on M is called $\mathrm{Vect}(M)$.

The naive idea of a vector field on $\mathbb{R}^m$ being a function $\alpha : \mathbb{R}^m \to \mathbb{R}^m$ translates into this picture by assigning to α the vector field $v_\alpha = \alpha^\mu \partial_\mu : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)$. In fact, the operators ∂_μ are a basis of $\mathrm{Vect}(\mathbb{R}^m)$ as a $C^\infty(\mathbb{R}^m)$-module.

A.1.2 Tangent and Cotangent Space

Definition A.3 (Tangent Space)**.** Let us now fix a point $p \in M$. Similar to the above definition, a *tangent vector* v_p is an $\mathbb{R}$-linear function $v_p : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz law

$$v_p(f \cdot g) = g(p)\, v_p(f) + f(p)\, v_p(g) \ . \tag{A.2}$$

The set of all tangent vectors at p is called the *tangent space* T_pM. It is a vector space over $\mathbb{R}$.

There is a correspondence between vector fields and tangent vectors: Each vector field v yields a tangent vector v_p by evaluation: $v_p(f) = v(f)|_p$. Additionally, two vector fields are equal if and only if the tangent vectors defined by them are equal at every point p.

Alternatively, we could have defined tangent vectors as equivalence classes of curves through that point. We can see this because we can assign to each curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ a tangent vector $\gamma'(0)$ via

$$\gamma'(0) : C^\infty(M) \to \mathbb{R}, f \mapsto (f \circ \gamma)'(0) \ . \tag{A.3}$$

Lemma A.4. *Those definitions are equivalent.*

Proof (taken from [56, Ch. 5B]). As we have seen, we can assign to each curve a tangent vector. We arrange the curves into equivalence classes in such a way that this map is injective. The only thing left to show is that we can find a curve γ for each v_p such that $v_p = \gamma'(0)$.

Without loss of generality, we work in a chart where $\varphi(p) = 0$. We set $h = f \circ \varphi^{-1}$, then[1] $h(x) = h(0) + \int_0^1 h_{,i}(tx)\ \mathrm{d}t \cdot x^i$. Using definition A.3, we get

$$v_p(f) = v_p[q \mapsto h(\varphi(q))] = h_{,i}(0) \cdot \underbrace{v_p[q \mapsto \varphi^i(q)]}_{c^i} \ .$$

Now we finally set $\tilde{\gamma}^i(t) = t \cdot c^i$ and $\gamma = \varphi^{-1} \circ \tilde{\gamma}$ such that $(f \circ \gamma)'(0) = (h \circ \tilde{\gamma})'(0) = h_{,i}(0)\, c^i$. □

Theorem A.5. *If we work on a chart (U, φ) of M we can write down the tangent vectors*

$$\left(\partial_\mu\right)_p : C^\infty(M) \to \mathbb{R}, f \mapsto \left.\partial_\mu(f \circ \varphi^{-1})\right|_{\varphi(p)} \ . \tag{A.4}$$

Those are a basis of T_pM. We will omit the $(\cdot)_p$ where it is obvious from the context.

Proof. We will show that those tangent vectors span T_pM. In lemma A.4, we have seen that every tangent vector can be written in the form $\gamma'(0)$. Now, according to the chain rule

$$\gamma'(0)\, f = (f \circ \gamma)'(0) = (f \circ \varphi^{-1} \circ \varphi \circ \gamma)'(0) = \left[\varphi^\mu \circ \gamma\right]'(0) \cdot \left(\partial_\mu\right)_{\gamma(0)} f \ .$$ □

Definition A.6 (Cotangent Space)**.** The *cotangent space* T_p^*M is the dual space of T_pM. In a chart, the basis dual to $\{\partial_\mu\}$ is written $\{\mathrm{d}x^\mu\}$, i.e. $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$.

We call the elements of T_p^*M *dual vectors* in p. To each $f \in C^\infty(M)$ we assign a dual vector, its *differential* in that point, $\left.\mathrm{d}f\right|_p = \left.\partial_\mu f\right|_p (\mathrm{d}x^\mu)_p$.

The definition of the differential of f ensures that for a tangent vector $v_p = v^\mu \left(\partial_\mu\right)_p$,

$$\left.\mathrm{d}f\right|_p (v_p) = v^\mu \left(\partial_\mu\right)_p f = v_p(f) \ . \tag{A.5}$$

Note. As we can see, the $\mathrm{d}x^\mu$ are the differentials of the coordinate functions $x^\mu : U \to \mathbb{R}, p \mapsto \varphi^\mu(p)$.

A.1.3 Tensors

Definition A.7 (Tensor)**.** An (r,s)-*tensor* T is an element of $T_pM^{\otimes r} \otimes T_p^*M^{\otimes s}$. That is, it is a multilinear map $T : (T_p^*M)^r \times (T_pM)^s \to \mathbb{R}$. In a chart, it can be expanded in the basis

$$T = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes \mathrm{d}x^{\nu_1} \otimes \dots \otimes \mathrm{d}x^{\nu_s} \ . \tag{A.6}$$

If there are two charts in a neighborhood of p we can do a *coordinate transformation* by changing from one description to the other. Let us call the coordinate functions of the one chart $x^\mu(p)$ and of the other $y^\mu(p)$. Slightly abusing the notation, we will call the transition functions $x(y)$ and $y(x)$. The basis transformation formulas then take the intuitive form

$$\mathrm{d}y^\mu = \frac{\partial y^\mu}{\partial x^\nu}\mathrm{d}x^\nu \qquad \text{and} \tag{A.7}$$

$$\partial_{y^\mu} = \frac{\partial}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu}\partial_{x^\nu} \ . \tag{A.8}$$

From this we derive how the components of tensors transform.

[1] $h_{,i}$ denotes the partial derivative, summation over the repeated "i" is implied.

Example A.8. Let $T = T^\mu{}_\nu \partial_{x^\mu} \otimes \mathrm{d}x^\nu$, this has to be equal to $T = (T')^\mu{}_\nu \partial_{y^\mu} \otimes \mathrm{d}y^\nu$. Then we get that the transformed components are

$$(T')^\mu{}_\nu = T^\rho{}_\lambda \frac{\partial y^\mu}{\partial x^\rho} \frac{\partial x^\lambda}{\partial y^\nu} \ . \tag{A.9}$$

Tensors of different ranks transform analogously.

Definition A.9 (Tensor Fields)**.** A tensor field is a smooth assignment of a tensor to each point. Smooth means here that the components are smooth functions from M to $\mathbb{R}$ in every chart.

An $(1,0)$-tensor field is a vector field in $\mathrm{Vect}(M)$ as discussed above. $(0,1)$-tensor fields are called *one-forms*, the set containing them is written $\Omega^1(M)$ in the context of differential forms (see subsection A.2). A $(0,0)$-tensor field is just an element of $C^\infty(M)$, in that context we write $\Omega^0(M)$ instead.

Note. Now we can also define the *differential* of $f \in C^\infty(M)$ as a one-form. In every chart, we want that $\mathrm{d}f = \partial_\mu f \ \mathrm{d}x^\mu$. According to the discussion after definition A.6, we use $\mathrm{d}f(v) = v(f)$ as the global definition (where v is a vector field).

A.1.4 Induced Maps

We consider now a smooth map $\Phi : M \to N$ between two differentiable manifolds M and N.

Definition A.10 (Smooth Map)**.** Φ is a *smooth map* if and only if $\psi \circ \Phi \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth for any charts (U, φ) on M and (V, ψ) on N. Further, Φ is a *diffeomorphism* if Φ is smooth and bijective, and also Φ^{-1} is smooth.

Definition A.11 (Pullback of Functions)**.** Let $f \in C^\infty(N)$ be a function, then we can define its *pullback via* Φ as

$$\Phi^* f : M \to \mathbb{R}, (\Phi^* f)(p) = (f \circ \Phi)(p) \ , \tag{A.10}$$

i.e. Φ^* is a map $C^\infty(N) \to C^\infty(M)$, or equivalently $\Omega^0(N) \to \Omega^0(M)$.

Definition A.12 (Pushforward of Tangent Vectors)**.** Let $v_p \in T_pM$ be a tangent vector of M in p. We can then define its *pushforward via* Φ, $\Phi_* v_p \in T_{\Phi(p)}N$, as

$$(\Phi_* v_p) f = v_p(\Phi^* f) \tag{A.11}$$

for all $f \in C^\infty(N)$, i.e. Φ_* is a map $T_pM \to T_{\Phi(p)}N$. Φ_* is also called the *differential* of Φ, written $\mathrm{d}\Phi$.

Lemma A.13. *If $N = \mathbb{R}$, this definition of the differential is the same as the other one given above.*

Proof. Φ_* maps T_pM to $T_{\Phi(p)}\mathbb{R} \cong \mathbb{R}$ as it should. Take $\xi \in C^\infty(\mathbb{R})$ and let $\Phi_*(v_p)$ act on ξ. The result is $\Phi_*(v_p)\xi = v_p(\Phi) \cdot \xi'(\Phi(p))$. □

Definition A.14 (Pushforward of Vector Fields)**.** We can only define the *pushforward of a vector field* if Φ is a diffeomorphism. In that case, we define

$$(\Phi_* v) : q \mapsto (\Phi_* v)_q = \Phi_* v_{\Phi^{-1}(q)} \tag{A.12}$$

for a vector field $v : p \mapsto v_p \in \mathrm{Vect}(M)$. Here we understand Φ_* as a map $\mathrm{Vect}(M) \to \mathrm{Vect}(N)$.

Note. If we let $v = v^\mu \partial_{x^\mu}$ and $\Phi_* v = (v')^\mu \partial_{y^\mu}$ like in the discussion before (A.9), then $(v')^\mu = v^\nu \frac{\partial y^\mu}{\partial x^\nu}$ like in (A.9). What we did above was a passive coordinate transformation, pushing forward corresponds to an active coordinate transformation.

Definition A.15 (Pullback of Cotangent Vectors)**.** Let $\omega \in T^*_{\Phi(p)}(N)$, then its *pullback via* Φ is

$$(\Phi^* \omega)(v) = \omega(\Phi_* v) \tag{A.13}$$

where $v \in T_pM$, i.e. $\Phi^* : T^*_{\Phi(p)}(N) \to T^*_pM$.

Definition A.16 (Pullback of One-Forms)**.** We can pull back one-forms without requiring Φ to be a diffeomorphism by simply defining

$$(\Phi^*\omega)_p = \Phi^*(\omega_{\Phi(p)}) \ . \tag{A.14}$$

Now Φ^* is a map $\Omega^1(N) \to \Omega^1(M)$.

Note. We can naturally define the pullback of all tensor fields of type $(0, s)$. The components of such tensor fields transform under the pullback as one would expect from (A.9).

Theorem A.17 (Naturalness)**.** *Let $f \in C^\infty(N)$ and $\Phi : M \to N$ smooth. Then*

$$\Phi^*(\mathrm{d}f) = \mathrm{d}(\Phi^* f) \ . \tag{A.15}$$

Proof. It suffices to show that both one-forms are equal in any point p. Let $v_p \in T_pM$, then

$$\left[\Phi^*(\mathrm{d}f)\right]\big|_p v_p = \left[\Phi^*(\mathrm{d}f|_{\Phi(p)})\right] v_p = \mathrm{d}f|_{\Phi(p)} (\Phi_* v_p) = (\Phi_* v_p)(f) = v_p(\Phi^* f) = \mathrm{d}(\Phi^* f)\big|_p v_p \ . \qquad \square$$

Example A.18. Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ be a curve and $\omega = \mathrm{d}x + \mathrm{d}y \in \Omega^1(\mathbb{R}^2)$. Then we can use theorem A.17 to pull back ω to I:

$$\gamma^*(\mathrm{d}x + \mathrm{d}y) = \mathrm{d}(\gamma^* x) + \mathrm{d}(\gamma^* y) = \mathrm{d}(\gamma^x(t)) + \mathrm{d}(\gamma^y(t)) = \left[(\gamma^x)'(t) + (\gamma^y)'(t)\right] \ \mathrm{d}t \ .$$

A.1.5 Metrics

Definition A.19 (Metric)**.** A *Riemannian Metric* g is a $(0, 2)$-tensor field which is symmetric ($g(v, w) = g(w, v)$ for $v, w \in \mathrm{Vect}(M)$) and positive definite ($g(v, v) \geq 0$ and $g(v, v) = 0$ iff $v = 0$).

g is *pseudo-Riemannian* if it is symmetric and non-degenerate (if $g(v, w) = 0$ for all w, then v must be 0).

The *inverse metric* is a $(2, 0)$-tensor field, also written as g, satisfying $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$.

The map $(v, w) \mapsto g(v, w)$ is a scalar product for the vector space T_pM. A scalar product always induces a canonical isomorphism between a vector space and its dual. This isomorphism is called the *musical isomorphism*.

Definition A.20 (Musical Isomorphisms)**.** For $v \in T_pM$ we define $v^\flat \in T_p^*M$ such that $v^\flat(w) = g(v, w)$ for all $w \in T_pM$. In components this means that

$$(v^\flat)_\mu = g_{\mu\nu} v^\nu \ . \tag{A.16}$$

Analogously, for $\omega \in T_p^*M$ we define $\omega^\sharp \in T_pM$ such that $g(\omega^\sharp, v) = \omega(v)$ for all $v \in T_pM$. In components:

$$(\omega^\sharp)^\mu = g^{\mu\nu} \omega_\nu \ . \tag{A.17}$$

This is called pulling indices up and down.

Example A.21. The pseudo-Riemannian metric $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ is called the *Minkowski metric*. It has *signature* $(3, 1)$ because there are 3 positive and 1 negative eigenvalues, also written as $s = 3$.

A.2 Differential Forms

A.2.1 Exterior Product

We will first define the exterior algebra of an arbitrary vector space V with basis $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$.

Definition A.22 (Two-Vectors)**.** Let $\boldsymbol{v}, \boldsymbol{w} \in V$ be vectors. We define their *exterior product* to be

$$\boldsymbol{v} \wedge \boldsymbol{w} = \boldsymbol{v} \otimes \boldsymbol{w} - \boldsymbol{w} \otimes \boldsymbol{v} \ . \tag{A.18}$$

$\boldsymbol{v} \wedge \boldsymbol{w}$ lies in the space $\Lambda^2 V$ of two-vectors. This space is spanned by the $\binom{n}{2}$ basis elements $\boldsymbol{e}_i \wedge \boldsymbol{e}_j$ $(1 \leq i < j \leq n)$.

The components of this two-vector are given by

$$(\boldsymbol{v} \wedge \boldsymbol{w})_{ij} = v_i w_j - v_j w_i \ . \tag{A.19}$$

They are antisymmetric by definition.

Note. The components of a two-vector are defined such that

$$\boldsymbol{v} \wedge \boldsymbol{w} = (\boldsymbol{v} \wedge \boldsymbol{w})_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{1}{2} (\boldsymbol{v} \wedge \boldsymbol{w})_{ij}\, \boldsymbol{e}_i \wedge \boldsymbol{e}_j = \sum_{i<j} (\boldsymbol{v} \wedge \boldsymbol{w})_{ij}\, \boldsymbol{e}_i \wedge \boldsymbol{e}_j \ . \tag{A.20}$$

Definition A.23 (q-Vectors)**.** In general, the space $\Lambda^q V$ of *q-vectors* is spanned by $\binom{n}{q}$ basis elements $\bigwedge_{k=1}^{q} \boldsymbol{e}_{i_k}$ $(1 \leq i_1 < \cdots < i_q \leq n)$, where $\bigwedge$ is the antisymmetrization[2] of the tensor product.

The space $\Lambda V = \bigoplus_{q=0}^{n} \Lambda^q V$ is called the *Grassmann Algebra* or *Exterior Algebra* over V.

Note. Analogously, we define the antisymmetric components $\omega_{i_1 \ldots i_q}$ of a q-vector ω to be such that

$$\omega = \omega_{i_1 \ldots i_q} \bigotimes_{k=1}^{q} \boldsymbol{e}_{i_k} = \frac{1}{q!}\, \omega_{i_1 \ldots i_q} \bigwedge_{k=1}^{q} \boldsymbol{e}_{i_k} \ . \tag{A.21}$$

We now want to define the exterior product of a q-vector ω and an r-vector η. We impose the following conditions: Firstly, ΛV should become an associative (graded) algebra together with the exterior product $\wedge$. Secondly, for q vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_q$ we want

$$\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_q = \bigwedge_{i=1}^{q} \boldsymbol{v}_i \ ,$$

where $\bigwedge$ was defined above as the antisymmetrization of the tensor product. Those conditions are uniquely satisfied by the following definition:

Definition A.24 (Exterior Product)**.** The components $(\omega \wedge \eta)_{i_1 \ldots i_{q+r}}$ are given by

$$(\omega \wedge \eta)_{i_1 \ldots i_{q+r}} = \frac{1}{q!r!} \omega_{[i_1 \ldots i_q} \eta_{i_{q+1} \ldots i_{q+r}]} = \frac{1}{q!r!} \sum_{\pi} (-1)^{\pi}\, \omega_{\pi(i_1) \cdots \pi(i_q)} \eta_{\pi(i_{q+1}) \cdots \pi(i_{q+r})} \ . \tag{A.22}$$

Proof. Obviously,

$$\omega \wedge \eta = \frac{1}{q!r!} \omega_{i_1 \ldots i_q} \eta_{i_{q+1} \ldots i_{q+r}}\, \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{q+r}} \ .$$

This can be compared with the definition

$$\omega \wedge \eta = \frac{1}{(q+r)!} (\omega \wedge \eta)_{i_1 \ldots i_{q+r}}\, \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{q+r}}$$

to read off the antisymmetric $(\omega \wedge \eta)_{i_1 \ldots i_{q+r}}$. □

Example A.25. If $\omega = \frac{1}{2} \omega_{ij}\, \boldsymbol{e}_i \wedge \boldsymbol{e}_j$ is a two-vector and $\eta = \eta_k \boldsymbol{e}_k$ is a vector, then $\xi = \omega \wedge \eta$ has components

$$\xi_{ijk} = \frac{1}{2} \left(\omega_{12}\eta_3 + \omega_{31}\eta_2 + \omega_{23}\eta_1 - \omega_{21}\eta_3 - \omega_{13}\eta_2 - \omega_{32}\eta_1 \right) = \omega_{12}\eta_3 + \omega_{31}\eta_2 + \omega_{23}\eta_1 \ .$$

[2] That means $\bigwedge_k \boldsymbol{v}_k = \sum_\pi (-1)^\pi \bigotimes_k \boldsymbol{v}_{\pi(k)}$, where the sum runs over all permutations π of the indices i.

Theorem A.26. *The Grassmann Algebra is a graded commutative algebra, i.e. for a q-vector ω and a r-vector η*

$$\omega \wedge \eta = (-1)^{qr}\, \eta \wedge \omega \ . \tag{A.23}$$

Proof. The theorem holds by definition for ω, η of the form $\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_q}$ and $\boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_r}$, respectively, and therefore it is true for all ω and η. □

Definition A.27 (Differential Form)**.** A *differential form* or *q-form* is a smooth assignment of an element of $\Lambda^q T_p^* M$ to each point p in a manifold M. The space of q-forms is written $\Omega^q(M)$. The space of all differential forms is $\Omega(M)$.

Note. Locally, a basis of $\Omega^q(M)$ (as a $C^\infty(M)$-module) is $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_q}$, $1 \leq \mu_1 < \cdots < \mu_q \leq m$.

A.2.2 Exterior Derivative

Definition A.28. We define the *exterior derivative* $\mathrm{d} : \Omega^q(M) \to \Omega^{q+1}(M)$ by the following properties:

i) $\mathrm{d} : \Omega^0(M) \to \Omega^1(M)$ is the differential as defined above.

ii) d is $\mathbb{R}$-linear.

iii) d satisfies a graded Leibniz identity: For $\omega \in \Omega^q(M)$ and $\eta \in \Omega^r(M)$,

$$\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^q \omega \wedge \mathrm{d}\eta \ . \tag{A.24}$$

iv) $\mathrm{d}(\mathrm{d}\omega) = 0$ for all differential forms ω.

Lemma A.29. d *is well-defined and unique. On a chart with* $\omega = \frac{1}{q!}\omega_{\mu_1 \ldots \mu_q}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_q}$,

$$\mathrm{d}\omega = \frac{1}{q!}\left(\partial_\nu \omega_{\mu_1 \ldots \mu_q}\right)\, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_q} \ . \tag{A.25}$$

Note. This means that $(\mathrm{d}\omega)_{\mu_1 \ldots \mu_{q+1}} = \frac{1}{q!}\partial_{[\mu_1}\omega_{\mu_2 \ldots \mu_{q+1}]}$.

Example A.30. Let $A = A_\mu \mathrm{d}x^\mu$ be a one-form and $F = \mathrm{d}A$. Then $F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{A.26}$$

Lemma A.31 (Naturalness)**.** *Theorem A.17 extends to $\omega \in \Omega^q(N)$ and $\Phi : M \to N$ smooth:*

$$\Phi^*(\mathrm{d}\omega) = \mathrm{d}(\Phi^*\omega) \ . \tag{A.27}$$

A.2.3 Hodge Star

Definition A.32 (Volume Form)**.** A *volume form* vol is a nowhere-vanishing element of $\Omega^m(M)$.
A manifold on which a volume form exists is called *orientable*.

If there is a metric given on an orientable manifold, there is a canonical volume form: On an orientable manifold, we can use oriented charts on which $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m = \rho$ vol with a positive function $\rho \in C^\infty(M)$. The transition functions between oriented charts are orientation-preserving, i.e. $\det \frac{\partial y^\mu}{\partial x^\nu} > 0$.

Now define on every chart the canonical volume form (for simplicity just called vol) as

$$\mathrm{vol} = \sqrt{|g|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \tag{A.28}$$

(g is the determinant of the matrix $(g_{\mu\nu})$). Because the transition functions are orientation-preserving, this definition agrees on the overlap of two charts.

Definition A.33 (Hodge Star)**.** Let's first define the *inner product* of two q-forms ω and η. It is a 0-form given by $\langle \omega, \eta \rangle = \frac{1}{q!} g^{\mu_1 \nu_1} \cdots g^{\mu_q \nu_q} \omega_{\mu_1 \dots \mu_q} \eta_{\nu_1 \dots \nu_q}$.

The *Hodge star* operator $*$ gives a canonical isomorphism between the spaces $\Omega^q(M)$ and $\Omega^{m-q}(M)$ of the same dimension. The Hodge star of a q-form ω is defined by the equation

$$\eta \wedge *\omega = \langle \eta, \omega \rangle \text{ vol} \tag{A.29}$$

for all $\eta \in \Omega^q(M)$. $*\omega \in \Omega^{m-q}(M)$ is called the Hodge dual of ω. Note that $*1 = \text{vol}$.

Lemma A.34 (Calculating the Hodge Dual)**.** *Let $\epsilon_{i_1 \dots i_m}$ be ± 1 as usual, and $\omega \in \Omega^q(M)$. Then*

$$(*\omega)_{\nu_1 \dots \nu_{m-q}} = \frac{\sqrt{|g|}}{q!} \omega_{\mu_1 \dots \mu_q} \epsilon^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_{m-q}} \ . \tag{A.30}$$

Proof. One can prove (A.29) from (A.30) for basis elements $\omega = \mathrm{d}x^{\mu_1} \wedge \cdots \mathrm{d}x^{\mu_q}$ and $\eta = \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_q}$ with fixed μ_i and ν_i. It follows that it is true for all ω and η. ☐

Lemma A.35 (Square)**.** *In a manifold of signature $(s, m-s)$,*

$$*^2 = (-1)^{q(m-q)+s} \ . \tag{A.31}$$

A.3 Homology and Cohomology

A.3.1 Chains and Boundaries

Definition A.36 (Simplex)**.** An *oriented r-simplex* in $\mathbb{R}^m$ is the convex hull of $r+1$ points $p_0, \dots, p_r \in \mathbb{R}^m$ which are affinely independent, that means that the vectors $p_1 - p_0, \dots, p_r - p_0$ are linearly independent. We write $\sigma_r = \langle p_0 \dots p_r \rangle$ to denote the convex hull

$$\left\{ p_0 + \sum_i t_i (p_i - p_0) : t_i \geq 0, \sum_i t_i \leq 1 \right\} \ . \tag{A.32}$$

A *singular r-simplex* in M is the image of σ_r under a smooth map $f : \sigma_r \to M$.

Definition A.37 (Chain Group)**.** We now take the free $\mathbb{R}$-module over the set of r-simplexes in M. The result is the *chain group* $C_r(M)$ consisting of elements of the form $\sum_i a_i s_i$, where $a_i \in \mathbb{R}$ and s_i is an r-simplex.

Definition A.38 (Boundary)**.** The boundary of a simplex $\sigma_r = \langle p_0 \dots p_r \rangle$ in $\mathbb{R}^m$ is given by the chain

$$\partial \sigma_r = \sum_{i=0}^{r} (-1)^i \langle p_0 \dots \hat{p}_i \dots p_r \rangle \in C_{r-1}(\mathbb{R}^m) \ . \tag{A.33}$$

If $f : \sigma_r \to M$ is a singular r-simplex, its boundary is[3] $f(\partial \sigma_r)$.

Theorem A.39 (Chain Complex)**.** *$(C(M), \partial)$ is a* chain complex*. That means that $\partial : C_r(M) \to C_{r-1}(M)$ are group homomorphisms and $\partial^2 = 0$.*

Proof. The maps ∂ are group homomorphisms by definition. Furthermore, applying ∂ again to (A.33) obviously gives zero. ☐

Note. This construction will lead us to *singular* homology groups of manifolds. A very similar construction yields the *simplicial* homology of topological spaces. For this, we consider simplicial complexes. Those are collections of simplexes in $\mathbb{R}^m$ the intersections of which are common faces. We can define a boundary operator in the simplicial complex in an obvious way.

[3] The notation means $\sum_{i=0}^{r} (-1)^i f(\langle p_0 \dots \hat{p}_i \dots p_r \rangle) \in C_{r-1}(M)$.

A simplicial complex triangulates a topological space X if there is a homeomorphism between the complex as a subset of $\mathbb{R}^m$ and X. For a topological space that has a triangulation we can then define its simplicial homology groups to be the homology groups of the simplicial complex (see below). They are isomorphic to the singular homology groups we will define below, this is a useful way for calculating the singular homology.

Note also that in the context of simplicial homology the chain groups are usually defined to be free abelian groups (i.e. $\mathbb{Z}$-modules) over the simplexes instead of $\mathbb{R}$-modules. For now we will only be concerned with real coefficients, but everything works equally fine with arbitrary coefficient rings.

A.3.2 Homology Groups

Definition A.40. For any chain complex (C, ∂) we can define *homology groups* as follows:

- The group of *r-cycles* Z_r is the kernel of $\partial : C_r \to C_{r-1}$.
- The group of *r-boundaries* B_r is the image of $\partial : C_{r+1} \to C_r$.
- B_r is a subgroup of Z_r because of $\partial^2 = 0$. The *r-th homology group* is given by

$$H_r = Z_r/B_r \ . \tag{A.34}$$

Definition A.41. The *singular homology groups* $H_r(M)$ of a manifold are the homology groups of its singular chain complex $(C(M), \partial)$.

The *Betti numbers* b_r of M are their dimensions: $b_r(M) = \dim H_r(M)$

Lemma A.42 (Basic Properties)**.**

i) "Dimension Axiom"*: $H_n(M) = 0$ for $n > 0$ if M consists of only one point.*

ii) "Additivity"*: If M is a disjoint union of path connected components M_i, then $H_r(M) = \bigoplus_i H_r(M_i)$.*

iii) $H_0(M) = \mathbb{R}$ if and only if M is path connected.

Proof. The first claim is obvious.

To show the second statement, one just has to see that all the groups C_n, Z_n and B_n split in such direct sums.

As for the third, because of the previous statement it suffices to show that $H_0(M) = \mathbb{R}$ if M is path connected. Any two points $p_1, p_2 \in Z_0(M)$ are equivalent, because they are the boundary of the path between them. Therefore $H_0(M)$ is one-dimensional. □

Note. An important concept in algebraic topology are the homology groups $H_r(M, A)$ of M relative to a subspace A. They are the homology groups of the quotient chain complex $C(M)/C(A)$.

Now we can list some more properties of homology, proven for example in [57]. These properties, together with the ones mentioned in lemma A.42, can be used to define homology axiomatically:

i) *"Homotopy Axiom"*: If two pairs (M, A) and (N, B) are homotopy equivalent, their homology groups are isomorphic.

ii) *"Excision"*: If $A \subset U \subset M$ with $\bar{A} \subset \operatorname{int} U$, then $H_n(M, U)$ and $H_n(M \backslash A, U \backslash A)$ are isomorphic.

iii) *"Mayer-Vietoris sequence"*: Let $U, V \subset X$ be subspaces. Under certain conditions (for example if $\operatorname{int} U \cup \operatorname{int} V = X$) there exists an exact sequence[4]:

$$\cdots \to H_{n+1}(X) \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to \dots \ .$$

A final extremely important property of homology is that for every short exact sequence $0 \to C' \to C \to \bar{C} \to 0$ of chain complexes, one can construct a long exact sequence in homology:

$$\cdots \to H_{n+1}(\bar{C}) \to H_n(C') \to H_n(C) \to H_n(\bar{C}) \to \cdots \ .$$

[4] A sequence $\cdots \xrightarrow{f_{n-1}} O_n \xrightarrow{f_n} O_{n+1} \xrightarrow{f_{n+1}} \cdots$ of objects and morphisms is called *exact* if for every two subsequent morphisms $\ker f_{n+1} = \operatorname{im} f_n$.

A.3.3 Integration of Differential Forms

There are two closely related notions of integrating differential forms. In the first case, we want to integrate a q-form over a q-chain:

Definition A.43. Let $\omega \in \Omega^q(M)$ and let $c \in C_q(M)$ be a simplex, i.e. there is a smooth map $f : \sigma_q \to c$. Then

$$\int_c \omega = \int_{\sigma_q} f^* \omega \tag{A.35}$$

(a q-form can be integrated over a submanifold of $\mathbb{R}^m$ in an obvious manner). For a general $c \in C_q(M)$, we extend the definition linearly.

The other case is the integration of a volume form over the whole manifold. If it is not possible to cover all of M with only one chart, we need to split the integral over the single charts.

Definition A.44. Let (U_i, φ_i) be an atlas of M such that every point of M is only in a finite number of charts (M needs to be paracompact, i.e. every open cover has a refinement that is locally finite). A collection of functions $\varepsilon_i : M \to [0,1]$ is called a *partition of unity* if the support of ε_i is within U_i and $\sum_i \varepsilon_i = 1$.

For $\omega \in \Omega^m(M)$, we can now define

$$\int_M \omega = \sum_i \int_{U_i} \varepsilon_i \wedge \omega = \sum_i \int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* (\varepsilon_i \wedge \omega) \ . \tag{A.36}$$

Now we can finally state Stokes' Theorem which generalizes the fundamental theorem of calculus and several identities from vector calculus:

Theorem A.45 (Stokes)**.** *Let $\omega \in \Omega^{q-1}(M)$ and $c \in C_q(M)$. Then*

$$\int_c \mathrm{d}\omega = \int_{\partial c} \omega \ . \tag{A.37}$$

Proof. It suffices to show this for a simplex c. Because of (A.35), it then suffices to show that the formula holds for a q-simplex in $\mathbb{R}^m$. We then note that it suffices to use a $(q-1)$-form ω of the type $\omega = g(\boldsymbol{x})\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{q-1}$. In this case, it is a simple calculation. □

A.3.4 Cohomology Groups

Definition A.46 (DeRham Cohomology)**.** The differential forms $\Omega(M)$ together with the exterior derivative d form a cochain[5] complex. The cohomology groups of it are called the *deRham cohomology* of M: $Z^q(M)$ is the set of *closed* q-forms ω with $\mathrm{d}\omega = 0$ and $B^q(M)$ is the set of *exact* q-forms ω with $\omega = \mathrm{d}\eta$. Then

$$H^q(M) = Z^q(M)/B^q(M) \ . \tag{A.38}$$

The numbers $b^q = \dim H^q(M)$ are also called *Betti numbers* (and are in fact equal to b_q as we will see below).

Example A.47. H^0 consists of those functions $f \in C^\infty(M)$ with $\mathrm{d}f = 0$, i.e. of the locally constant functions. b^0 is therefore equal to the number of connected components of M (like b_0).

Theorem A.48 (DeRham Duality)**.** *H^q is the dual space of H_q.*

Proof. Let us define the map

$$(\cdot,\cdot) : C_q \times \Omega^q \to \mathbb{R}, (c, \omega) = \int_c \omega \ . \tag{A.39}$$

[5] It is a *co*chain complex because the coboundary operator d doesn't map $C^n \to C^{n-1}$ but $C^n \to C^{n+1}$. Cocycles, coboundaries and cohomology groups are defined in complete analogy to definition A.40.

Because of Stokes' theorem, $(\cdot,\cdot)$ also gives a well-defined bilinear map $H_q \times H^q \to \mathbb{R}$. It remains to prove that H_q and H^q are finite vector spaces and $(\cdot,\cdot)$ is non-degenerate. This is in fact the content of deRham's theorem, see [54, Thm. 6.2].

Now we can understand elements of H_q as linear maps from H^q to $\mathbb{R}$ and vice versa, this is the definition of the dual space. Note that there is no canonical isomorphism between the two spaces if we don't specify a scalar product. □

Theorem A.49 (Poincaré Duality)**.** *H_q and H^{m-q} are isomorphic, the isomorphism is called* Poincaré duality*. Especially,*

$$b_q = b^q = b_{m-q} = b^{m-q} . \tag{A.40}$$

Proof. Let us define the map

$$\Lambda : H^q \times H^{m-q} \to \mathbb{R}, \Lambda([\omega],[\eta]) = \int_M \omega \wedge \eta \tag{A.41}$$

(where $[\omega]$ is the equivalence class of ω in H^q). Again, this is well-defined, bilinear and non-degenerate. Therefore, H^{m-q} is isomorphic to the dual space of H^q which is H_q. □

How does the isomorphism look explicitly? In general a non-singular bilinear $\Lambda : V \times W \to \mathbb{R}$ gives an isomorphism $\varphi : V^* \to W$, where $\varphi(v')$ satisfies $\Lambda(v, \varphi(v')) = v'(v)$ for all $v \in V$. Therefore, the Poincaré dual of a $[c] \in H_q(M)$ is a class $[\gamma] \in H^{m-q}(M)$ such that

$$\int_c \omega = \int_M \omega \wedge \gamma \quad \forall \omega . \tag{A.42}$$

Definition A.50 (Pullback)**.** We define the pullback of a cohomology class to be

$$\Phi^*[\omega] = [\Phi^*\omega] . \tag{A.43}$$

This is well defined because of lemma A.31.

A.3.5 Hodge Theory

Using the Hodge star operator, we can define an inner product on $\Omega^q(M)$. Note that this doesn't give an inner product on $H^q(M)$, as $*$ and d are not compatible in general. In the following, let M be a compact orientable manifold.

Definition A.51 (Scalar Product)**.** Let $\omega, \eta \in \Omega^q(M)$, we define

$$(\omega,\eta) = \int_M \langle\omega,\eta\rangle \ \mathrm{vol} = \int_M \omega \wedge *\eta . \tag{A.44}$$

Definition A.52. The *adjoint exterior derivative* $\mathrm{d}^\dagger$ is defined by

$$(\mathrm{d}\omega,\eta) = (\omega, \mathrm{d}^\dagger\eta) \tag{A.45}$$

for all $\omega \in \Omega^q(M)$ and $\eta \in \Omega^{q-1}(M)$.

The *Laplacian* Δ is $\Delta = (\mathrm{d} + \mathrm{d}^\dagger)^2 = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}$.

Lemma A.53 (Adjoint Derivative)**.** *Obviously,* $\left(\mathrm{d}^\dagger\right)^2 = 0$.

Furthermore, the adjoint exterior derivative can be calculated as

$$\mathrm{d}^\dagger = (-1)^{m(q+1)+s+1} * \mathrm{d} * \tag{A.46}$$

on a manifold of signature s.

Example A.54. Electrodynamics is described by the electromagnetic field which is an exact two-form $F = \mathrm{d}A$ in Minkowski space. The homogeneous Maxwell equations follow directly from $\mathrm{d}F = \mathrm{d}^2 A = 0$. The inhomogeneous Maxwell equations can be written as $*\mathrm{d}{*F} = J$, or

$$\mathrm{d}^\dagger F = J \ . \tag{A.47}$$

Example A.55. The Laplacian of $f \in \Omega^0(M) = C^\infty(M)$ can be calculated as

$$\Delta f = (-1)^{s+1} {*\mathrm{d}*} \left(\partial_\mu f \ \mathrm{d}x^\mu\right) = \cdots = \frac{(-1)^{s+1}}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} g^{\nu\mu} \partial_\mu f\right) \ . \tag{A.48}$$

Finally, noticing the equality

$$(\omega, \Delta\omega) = (\mathrm{d}\omega, \mathrm{d}\omega) + (\mathrm{d}^\dagger\omega, \mathrm{d}^\dagger\omega) \geq 0 \tag{A.49}$$

for M of signature $s = 0$ (otherwise the scalar product is not positive definite) leads to a number of important theorems:

Definition A.56. A q-form ω is *(co-)closed* if $\mathrm{d}\omega = 0$ ($\mathrm{d}^\dagger\omega = 0$), *(co-)exact* if $\omega = \mathrm{d}\eta$ ($\omega = \mathrm{d}^\dagger\eta$) and *harmonic* if $\Delta\omega = 0$.

Lemma A.57. *ω is harmonic if and only if it is closed and coclosed.*

Theorem A.58 (Hodge Decomposition Theorem)**.** *Every ω can be written uniquely as the sum of an exact, a coexact and a harmonic form:*

$$\Omega^q(M) = \mathrm{d}\Omega^{q-1}(M) \oplus \mathrm{d}^\dagger\Omega^{q+1}(M) \oplus \mathrm{Harm}^q(M) \ . \tag{A.50}$$

Proof. The decomposition is possible because[6] every $\psi \perp \mathrm{Harm}^q(M)$ can be written as $\psi = \Delta\eta$ for some η. It is unique because $\mathrm{d}\Omega^{q-1}(M)$ and $\mathrm{d}^\dagger\Omega^{q+1}(M)$ are perpendicular, and additionally $\Delta\eta \perp \mathrm{Harm}^q(M)$ for all η. □

Theorem A.59 (Hodge's Theorem)**.** *Under the conditions as above (M is compact, orientable and Riemannian),*

$$H^q(M) \cong \mathrm{Harm}^q(M) \ . \tag{A.51}$$

Proof. Follows from theorem A.58 and lemma A.57. □

A.4 Fiber Bundles

A.4.1 Fiber Bundles and Structure Groups

Definition A.60 (Fiber Bundle)**.** A *fiber bundle* consists of the following data:

i) A differentiable manifold M, called *base space.*

ii) A differentiable manifold F, called the *fiber.*

iii) A differentiable manifold E, called *total space*, together with a *projection* $\pi : E \to M$.

iv) A *local trivialization*: For each $p \in M$ there has to be an open *chart* $U \subset M$ containing p such that $E|_U = \{q \in E : \pi(q) \in U\}$ and $U \times F$ are diffeomorphic as fiber bundles (see below).

We write $\pi : E \to M$ or $F \to E \xrightarrow{\pi} M$ for the fiber bundle.

Example A.61. For a given base space M and fiber F, the *trivial bundle* $M \times F$ is a fiber bundle.

[6] According to [54, Ex. 7.23], the proof of this is highly technical.

Definition A.62 (Bundle Morphism)**.** A bundle morphism between two bundles $\pi : E \to M$ and $\tilde{\pi} : \tilde{E} \to \tilde{M}$ is a smooth map $\Phi : E \to \tilde{E}$ together with a smooth map $\Psi : M \to \tilde{M}$ such that $\tilde{\pi} \circ \Phi = \Psi \circ \pi$.

If Φ and Ψ are diffeomorphisms, then the bundles are diffeomorphic.

Definition A.63 (Section)**.** A *section* s of a fiber bundle $F \to E \xrightarrow{\pi} M$ is a map $s : M \to E$ with $\pi \circ s = \mathrm{id}_M$. The space of sections is denoted by $\Gamma(E)$.

Definition A.64 (G-Bundle)**.** Let G be a group acting on the fiber of a fiber bundle $F \to E \xrightarrow{\pi} M$ from the left. Consider two charts $U_i, U_j \subset M$ with the bundle diffeomorphisms $\varphi_i : U_i \times F \to E|_{U_i}$ and $\varphi_j : U_j \times F \to E|_{U_j}$. The local trivialization is called a *G-atlas* if the transition functions $\varphi_i^{-1} \circ \varphi_j$ can be written as $(p, f) \mapsto (p, g_{ij}(p).f)$ for a $g_{ij} : U_i \cap U_j \to G$ where they are defined.

A *G-bundle* is a fiber bundle with an equivalence class[7] of G-atlases. G is called its structure group.

Note. We will use the notation $[p, f]_i = \varphi_i(p, f)$. The condition for the transition function then reads

$$[p, f]_j = [p, g_{ij}(p).f]_i \; . \tag{A.52}$$

Theorem A.65 (Fiber Bundle Construction Theorem)**.** *A G-bundle is uniquely specified by giving the base space, the fiber, charts U_i and functions $g_{ij} : U_i \cap U_j \to G$. The g_{ij} have to satisfy $g_{ii} \equiv e$, $g_{ij}g_{ji} \equiv e$ and the* cocycle condition *$g_{ij}g_{jk}g_{ki} \equiv e$.*

Proof. We let E be the union $\bigcup_i U_i \times F$ with identifications according to the g_{ij}. For details see [54, Ch. 9.2.2]. □

A.4.2 Vector Bundles and Principal Bundles

Definition A.66 (Vector Bundle)**.** A *vector bundle* is a fiber bundle where the fiber F is a vector space. Furthermore, the trivialization morphisms $U \times F \to E|_U$ have to be vector bundle morphisms.

A *vector bundle morphism* is a bundle morphism whose restriction to any fiber is linear.

Example A.67. A vector field is a section of the *tangent bundle* $TM = \bigcup_p T_pM$. The tangent bundle is a vector $\mathrm{GL}(\dim F)$-bundle, the transition functions between charts are $(V')^\mu = V^\nu \frac{\partial y^\mu}{\partial x^\nu}$ as seen in (A.9). In other words, the group element $g(p)$ acting from the left is the matrix $\left.\frac{\partial y^\mu}{\partial x^\nu}\right|_p$.

Example A.68. A function (0-form) is a section of the trivial bundle $M \times \mathbb{R}$.

A one-form is a section of the *cotangent bundle* $T^*M = \bigcup_p T_p^*M$.

More generally, differential forms are sections of the *form bundle* $\Lambda T^*M = \bigcup_p \Lambda T_p^*M$.

Note. Each vector bundle admits a global section, the zero section $s : M \to 0 \in F$.

Definition A.69. From given vector bundles we can create new vector bundles in several ways:

- Given a vector G-bundle $F \to E \xrightarrow{\pi} M$, we use the construction theorem A.65 to create the *dual bundle*, a vector G-bundle with base space M and fiber F^* dual to F. We reuse the transition functions g_{ij}, but they now act in the dual representation[8] on F^*.
- Given $\pi : E \to M$ and $\tilde{\pi} : \tilde{E} \to \tilde{M}$, we can define the *product bundle* in a pretty straightforward way:
$$(\pi \times \tilde{\pi}) : E \times \tilde{E} \to M \times \tilde{M}, (p, \tilde{p}) \mapsto (\pi(p), \tilde{\pi}(\tilde{p})) \; .$$
- Let $f : N \to M$ be a smooth map and $F \to E \xrightarrow{\pi} M$ a vector bundle. Then the *pull-back bundle* f^*E is constructed as a vector bundle over N with fiber F and transition functions $t_{ij} \circ f$. Alternatively it can be described as $\{(p, u) \in N \times E : f(p) = \pi(u)\}$.

[7] Two G-atlases are equivalent if their union is a G-atlas as well.

[8] Let the action of G on V be $g : V \to V, v \mapsto \rho(g)v$. Then the dual representation is the action $g : V^* \to V^*, \varphi \mapsto \rho^*(g)\varphi$ where $[\rho^*(g)\varphi]v = \varphi[\rho(g)^{-1}v]$. In components, $\rho^*(g) = \rho(g^{-1})^T$.

- Given vector G-bundles $F \to E \xrightarrow{\pi} M$ and $\tilde{F} \to \tilde{E} \xrightarrow{\tilde{\pi}} M$, the *Whitney sum bundle* is a vector G-bundle with base space M and fiber $F \oplus \tilde{F}$. One way to define it is as the pullback of the product bundle (using $f : M \to M \times M$ with $f(p) = (p, p)$). Another way is to just use theorem A.65 again, the transition functions are the direct sum of the transition functions of E and $\tilde{E}$.

- In the same setting, the *tensor product bundle* is a vector G-bundle with base space M and fiber $F \otimes \tilde{F}$.

Definition A.70 (Principal G-Bundle)**.** A *principal G-bundle* $P(M, G)$ is a G-bundle over the base M which has G as its fiber.

We can define a *right action* of G on $P(M, G)$. In a chart U_i, $u \in P(M, G)$ is written as $[\pi(u), g_i(u)]_i$. $h \in G$ acts from the right as

$$[\pi(u), g_i(u)]_i.h = [\pi(u), g_i(u)h]_i \tag{A.53}$$

Note. This is independent of the chart, because the transition functions $g_{ij} = g_i g_j^{-1}$ act from the left.

In more detail, let $u \in U_i \cap U_j$. Because of the G-bundle condition, $g_i(u) = g_{ij}(\pi(u))g_j(u)$. Thus

$$[\pi, g_i h]_i = [\pi, g_{ij} g_j h]_i = [\pi, g_{ji} g_{ij} g_j h]_j = [\pi, g_j h]_j \; .$$

Lemma A.71. *We can use a section of a principal bundle to choose which group element gets mapped to the unit element in a trivialization. Therefore, sections in charts correspond to local trivializations, and a principal bundle is trivial if and only if there is a global section.*

Definition A.72 (Associated Bundles)**.** Given a principal bundle $P(M, G)$ with G acting on a vector space F in a representation ρ, we can construct the *associated vector bundle* $P \times_\rho F$. It is a vector bundle with base space M and fiber F and can be constructed by identifying the points

$$(u, v) \sim (u.g, \rho(g^{-1})v) \tag{A.54}$$

in $P(M, G) \times F$.

We write an equivalence class as $[u, v]$ (warning: do not confuse with the local trivialization $[p, v]_i$).

The projection of the associated vector bundle is $\pi([u, v]) = \pi(u)$.

Note. On the other hand, given a vector G-bundle $\pi : E \to M$ there is an associated principal G-bundle, which we can construct using theorem A.65 and the transitions functions g_{ij} of E.

Definition A.73. A local trivialization $[\cdot, \cdot]_i$ of $P(M, G)$ canonically induces a local trivialization of the associated vector bundle which we will call $\{\cdot, \cdot\}_i$: We define

$$\{p, v\}_i = [[p, e]_i, v] \; . \tag{A.55}$$

Note that every element of $P \times_\rho F$ can be written in the form $[[p, e]_i, v]$ because $[u, v] = [[\pi(u), g_i(u)]_i, v] = [[\pi(u), e]_i, \rho(g_i(u))v]$.

Because $\{p, v\}_j = \{p, \rho(g_{ij})v\}_i$ we can see that $P \times_\rho G$ is a G-bundle.

A.4.3 Gauge Transformations

Definition A.74 (Gauge Transformation)**.** A *gauge transformation* is a diffeomorphism $\Phi : P \to P$ which maps fibers to fibers ($\pi \circ \Phi = \pi$) and is compatible with the right action: $\Phi(u.g) = \Phi(u).g$.

In a chart U_i with $u = [\pi(u), g_i(u)]_i$, we can describe the gauge transformation as $\bar{\Phi}_i : P|_{U_i} \to G$ acting from the left: $\Phi(u) = [\pi(u), \bar{\Phi}_i(u)g_i(u)]_i$. This means that $\bar{\Phi}_i(u) = g_i(\Phi u)g_i(u)^{-1}$. Because $\bar{\Phi}_i(u.g) = \bar{\Phi}_i(u)$, this induces a map $\Phi_i : U_i \to G$ such that

$$\Phi \, [\pi(u), g_i(u)]_i = [\pi(u), \Phi_i(\pi(u)) \, g_i(u)]_i \; . \tag{A.56}$$

Comparing this with the action in another chart U_j finally yields

$$\Phi_j(p) = g_{ij}^{-1}(p)\Phi_i(p)g_{ij}(p) \; . \tag{A.57}$$

Definition A.75. Gauge transformations act naturally on associated vector bundles by

$$\Phi\,[u,v] = [\Phi(u), v]\ . \tag{A.58}$$

This is well-defined thanks to $\Phi(u.g) = \Phi(u).g$.

In a chart, this means

$$\Phi\,\{p,v\}_i = [\Phi\,[p,e]_i, v] = [[p,\Phi_i(p)]_i, v] = \{p, \rho(\Phi_i(p))\,v\}_i\ . \tag{A.59}$$

A.5 Connections

A.5.1 Connections on Vector Bundles

Definition A.76 (Connection)**.** Let $\pi : E \to M$ be a vector bundle. A *connection* D on M assigns to each vector field $v \in \mathrm{Vect}(M)$ a map $D_v : \Gamma(E) \to \Gamma(E)$ with the following properties:

i) D is $C^\infty(M)$-linear in v, i.e. $D_{fv+gw} = fD_v + gD_w$ for functions $f, g \in C^\infty(M)$.

ii) For every v, D_v is $\mathbb{R}$-linear.

iii) Leibniz Rule: For $f \in C^\infty(M)$ and $s \in \Gamma(E)$,

$$D_v(fs) = v(f)s + fD_v s\ . \tag{A.60}$$

The object D_v is called *covariant derivative* in direction v. We use the abbreviation $D_\mu = D_{\partial_\mu}$.

A closely related notion is that of a *vector potential.* It appears when there is a canonical "best" connection D^0 on the bundle, we call it the *flat connection.* For example, after choosing a local trivialization we can on a chart U choose a basis $\{e_a\}$ of sections such that every section $s \in \Gamma(E|_U)$ can be written uniquely as $s = s^a e_a$, $s^a \in C^\infty(M)$. Then we define (in that chart)

$$D^0_\mu s = (\partial_\mu s^a)\, e_a\ . \tag{A.61}$$

Definition A.77 (Vector Potential)**.** A *vector potential* is an endomorphism valued one-form, i.e. a section of

$$\mathrm{End}(E) \otimes T^*M\ . \tag{A.62}$$

The *endomorphism bundle* $\mathrm{End}(E)$ is the tensor product of the bundle and its dual, $\mathrm{End}(E) = E \otimes E^*$. This means, a vector potential A has three indices: $A = A^a_{\mu b}\, e_a \otimes e^b \otimes \mathrm{d}x^\mu$.

Lemma A.78. *Any connection D can be written as the sum of the flat connection D^0 and a vector potential. On the other hand, for any vector potential A, $D = D^0 + A$ defines a connection.*

Proof. For the first part we need to show that $A(v)(s)$ defined as $D_v(s) - D^0_v(s)$ is $C^\infty(M)$-linear in v and s. Linearity in s follows because the $v(f)$-terms of the Leibniz rule cancel out.

For the second part we mainly have to check that $D^0 + A$ obeys the Leibniz rule:

$$D_v(fs) = D^0_v(fs) + A(v)(fs) = v(f)s + fD^0_v(s) + fA(v)s = v(f)s + fD_v(s)\ . \qquad \square$$

Definition A.79 (G-connection)**.** On a G-bundle $\pi : E \to M$ with G acting in the representation $\rho : G \to \mathrm{End}(E)$, we also have a representation $\mathrm{d}\rho : \mathfrak{g} \to \mathrm{End}(E)$ of the Lie algebra. We can consequently view $\mathfrak{g}$ as a subspace of $\mathrm{End}(E)$.

A connection D is a G-*connection* if in local coordinates the A_μ take values in $\mathfrak{g}$.

Note. Consider a chart U with a basis $\{e_a\}$ of sections. The equation $D = D^0 + A$ can then be written as

$$(D_\mu s)^a = \partial_\mu s^a + s^b A^a_{\mu b} \; . \tag{A.63}$$

We can use this to calculate the components of A: Because $D_\mu s = D_\mu(s^a e_a) = (\partial_\mu s^a)\, e_a + s^a D_\mu e_a$, we immediately get

$$D_\mu e_b = A^a_{\mu b} e_a \; . \tag{A.64}$$

So far, D and A are only defined locally, and the components of A depend on the basis of local sections. Let us consider another chart with basis $\{e'_a\}$ such that $e'_a = R^b_a e_b$ where both are defined (for a G-valued function R). Then A and A' need to satisfy the *compatibility condition*

$$A' = R^{-1} A R + R^{-1} \mathrm{d}R \; . \tag{A.65}$$

Definition A.80 (Levi-Civita Connection)**.** Let g be a pseudo-Riemannian metric on a manifold M. The *Levi-Civita connection* ∇ is the unique connection on TM which is *metric preserving* ($v[g(u,w)] = g(\nabla_v u, w) + g(u, \nabla_v w)$) and *torsion free* ($[v,w] = \nabla_v w - \nabla_w v$).

The vector potential is called the *Christoffel symbols* $\Gamma^\lambda_{\mu\nu}$. Note that all three indices are of the same type in this case. Torsion-freeness is equivalent to $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$.

Note. The condition that ∇ is metric preserving can be written as $\nabla_v g = 0$, using notation we will introduce in definition A.85.

Lemma A.81. *The Christoffel symbols can be calculated via*

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\eta} \left(\partial_\mu g_{\nu\eta} + \partial_\nu g_{\mu\eta} - \partial_\eta g_{\mu\nu}\right) \; . \tag{A.66}$$

A.5.2 Note on Connections on Principal Bundles

There's a different way of defining connections on principal and associated bundles. For $u \in P(M,G)$, we first define the *vertical subspace* $V_u P$ as the image of the map

$$\sharp : \mathfrak{g} \to T_u P, A \mapsto A^\sharp \quad \text{with} \quad A^\sharp : t \mapsto u.\exp(tA) \tag{A.67}$$

(using lemma A.4 to identify curves with tangent vectors).

Then a connection is defined to be a separation of $T_u P$ into $H_u P \oplus V_u P$ with[9] $H_{u.g} P = (R_g)_* H_u P$ and some smoothness condition. $H_u P$ is called the horizontal subspace. Equivalently, the connection is given by a one-form $\omega \in \mathfrak{g} \otimes T^* P$ called *Ehresmann connection*. ω has to satisfy $\omega(A^\sharp) = A$[10] and $(R_g)^* \omega_{u.g}(X) = g^{-1} \omega_u(X) g$. The definitions are equivalent because ω defines a projection of $T_u P$ on $V_u P$, i.e. $H_u P = \ker \omega = \{X \in T_u P : \omega(X) = 0\}$.

One more important ingredient is the notion of *horizontal lift*: If $\gamma : (-\varepsilon, \varepsilon) \to M$ is a curve, then there is a unique horizontal lift $\tilde\gamma : (-\varepsilon, \varepsilon) \to P$ with $\pi \circ \tilde\gamma = \gamma$, $\tilde\gamma'(t) \in H_{\tilde\gamma(t)} P$ and an initial condition $\tilde\gamma(0) = u_0$ (for an arbitrary u_0 with $\pi(u_0) = \gamma(0)$).

Now, we can finally use all this to define a connection on the associated vector bundle $P \times_\rho F$. Let $p \in M$, $X \in T_p M$ and $s \in \Gamma(P \times_\rho F)$. Choose a curve γ such that $\gamma(0) = p$ and $\gamma'(0) = X$. Along the curve, $s(\gamma(t)) = [\tilde\gamma(t), \eta(t)]$ for some lift $\tilde\gamma$ and a curve η in F. We define

$$D_X(s)\big|_p = [\tilde\gamma(0), \eta'(0)] \tag{A.68}$$

(which is independent of the choice of γ and the choice of initial condition for $\tilde\gamma$) and we define $D_v(s)$ for a vector field v by evaluating at each point $D_{v(p)}(s)$.

Lemma A.82. *The object D we just defined is a G-connection on the associated vector bundle (in the sense of our definitions above). On the other hand, every G-connection on a vector G-bundle can be understood in this way.*

[9] $R_g : P \to P$ is the right action of $g \in G$ on P.
[10] $\sharp$ can be made into a map $\sharp : \mathfrak{g} \to \mathrm{Vect}(P)$ in an obvious manner.

Proof. We will just show the most important point, that equation (A.64) holds.

Let p, X, γ and $\tilde{\gamma}$ be like in definition (A.68). Let furthermore p be in a chart U_i and let $\sigma_i : q \mapsto [q, e]_i$ be the section corresponding to the local trivialization (see lemma A.71). We choose a local basis of sections of the associated bundles by setting $e_a(q) = \{q, \boldsymbol{e}_a\}_i$, where $\{\boldsymbol{e}_a\}$ is a basis of the vector space F. We can write $\tilde{\gamma}(t) = [\gamma(t), g(t)]_i$. Without proof we accept the fact [54, (10.13)]

$$g'(0)\, g(0)^{-1} = -\sigma_i^* \omega(X) \ .$$

We want to calculate $D_X e_a\big|_p$. From the definition $e_a(\gamma(t)) = [\tilde{\gamma}(t), \eta(t)]$ we read off $\eta(t) = g(t)^{-1}.\boldsymbol{e}_a$. Now with (A.68),

$$\begin{aligned} D_X e_a\big|_p &= [\tilde{\gamma}(0), \frac{\mathrm{d}}{\mathrm{d}t} g(t)^{-1}\bigg|_{t=0} .\boldsymbol{e}_a] = \{p, g(0)\, \frac{\mathrm{d}}{\mathrm{d}t} g(t)^{-1}\bigg|_{t=0} .\boldsymbol{e}_a\}_i \\ &= \{p, -g'(0)\, g(0)^{-1}.\boldsymbol{e}_a\}_i = \{p, \sigma_i^*\omega(X).\boldsymbol{e}_a\}_i \ . \end{aligned}$$

$A = \sigma_i^* \omega$ is a $\mathfrak{g}$-valued vector potential. It is given only locally, but can be shown to satisfy (A.65) (see [54, (10.9)], therefore it defines a vector potential globally. This proves our claim. □

Lemma A.83. *A connection on an associated vector bundle is compatible with a gauge transformation* $\Phi : P \to P$*:*

$$\Phi(D_\mu s) = D_\mu^\Phi \Phi(s) \ , \tag{A.69}$$

where D^Φ *is the gauge transformed connection associated with* $H^\Phi_{\Phi(u)} P = \Phi_* H_u P$.

The transformed Ehresmann connection is $\omega^\Phi = (\Phi^{-1})^* \omega$ *and the transformed vector potential is*

$$A_i^\Phi = \Phi_i A_i \Phi_i^{-1} + \Phi_i \mathrm{d}\left(\Phi_i^{-1}\right) \ . \tag{A.70}$$

Proof. (A.69) follows from the definition (A.68): If we want to calculate $D_X^\Phi \Phi(s)\big|_p$, we can use the lift $\tilde{\gamma}^\Phi = \Phi \circ \gamma$ because then $\left(\tilde{\gamma}^\Phi\right)'(0) = \Phi_* \tilde{\gamma}'(0)$.

For the rest, see [55]. □

A.5.3 Derivatives and Curvature

Let $\pi : E \to M$ be a vector bundle and D a connection on it. In this subsection we will be concerned with *E-valued q-forms*, meaning sections of $E \otimes \Lambda^q T^* M$. We first need some technical definitions. For those note that a generic E-valued q-form is the sum of terms $s \otimes \omega$, where s is a section of E and ω is a q-form on M. We are only going to consider forms of this special type, the definitions for general forms are an obvious extension.

Definition A.84.

i) Assume there is some product between sections of E and those of another vector bundle $\tilde{E}$. For example, we have a canonical product between sections of $\mathrm{End}(E)$ and E, or between two sections of $\mathrm{End}(E)$. Then we can define a *wedge product* between an E-valued q-form $s \otimes \omega$ and an $\tilde{E}$-valued r-form $t \otimes \eta$:

$$(s \otimes \omega) \wedge (t \otimes \eta) = (s \cdot t) \otimes (\omega \wedge \eta) \ . \tag{A.71}$$

ii) The *graded commutator* of an $\mathrm{End}(E)$-valued q-form A and an $\mathrm{End}(E)$-valued r-form B is

$$[A, B] = A \wedge B - (-1)^{qr} B \wedge A \ . \tag{A.72}$$

Definition A.85. From our connection D on E we can derive connections on other bundles:

i) We define D^* on E^* by the requirement $v(\lambda \wedge s) = (D_v^* \lambda) \wedge s + \lambda \wedge (D_v s)$ for any vector field v, where λ is a section of E^* and s one of E.

ii) On $E \oplus \tilde{E}$, we can simply let $(D \oplus \tilde{D})_v(s, \tilde{s}) = (D_v s, \tilde{D}_v \tilde{s})$.

iii) And on $E \otimes \tilde{E}$, $(D \otimes \tilde{D})_v(s \otimes \tilde{s}) = D_v s \otimes \tilde{s} + s \otimes \tilde{D}_v \tilde{s}$.

iv) This gives us a connection on $\operatorname{End}(E) = E \otimes E^*$: Let A be a section of $\operatorname{End}(E)$ and s one of E. Then, by straightforward calculation, $(D_v A)s = D_v(As) - A(D_v s)$ or

$$D_v A = [D_v, A] \; . \tag{A.73}$$

Definition A.86 (Exterior Covariant Derivative)**.** The *exterior covariant derivative* of a form $s \otimes \eta$ is

$$\mathrm{d}_D(s \otimes \eta) = (\mathrm{d}_D s) \wedge \eta + s \otimes (\mathrm{d}\eta) \; . \tag{A.74}$$

The exterior covariant derivative of a section is defined over $(\mathrm{d}_D s)(v) = D_v s$, generalizing the formula $(\mathrm{d}f)(v) = v(f)$. In local coordinates, that means $\mathrm{d}_D s = D_\mu s \, \mathrm{d}x^\mu$.

Lemma A.87. *In local coordinates, we can write $D = D_0 + A$. Obviously, $\mathrm{d}_{D_0} = \mathrm{d}$.*
Let ω be an E-valued form and B an $\operatorname{End}(E)$*-valued form. Calculation shows*

$$\mathrm{d}_D\omega = \mathrm{d}\omega + A \wedge \omega \qquad \textit{and} \qquad \mathrm{d}_D B = \mathrm{d}B + [A, B] \; . \tag{A.75}$$

Definition A.88 (Curvature)**.** Let $\pi : E \to M$ be a vector bundle and D a connection on it. For two vector fields $v, w \in \operatorname{Vect}(M)$, the *curvature* $F(v, w)$ is an operator $\Gamma(E) \to \Gamma(E)$ given by

$$F(v, w) = [D_v, D_w] - D_{[v,w]} \; . \tag{A.76}$$

Lemma A.89.

i) F is $C^\infty(M)$-linear in v, w and in $s \in \Gamma(E)$.

ii) Hence, in local coordinates $F(v, w) = v^\mu w^\nu F_{\mu\nu}$ for $F_{\mu\nu} = [D_\mu, D_\nu]$.

iii) For a local basis of sections $\{e_a\}$, define $F_{\mu\nu} e_a = F^b_{\mu\nu\,a} e_b$. Then

$$F^b_{\mu\nu\,a} = \partial_\mu A^b_{\nu a} - \partial_\nu A^b_{\mu a} + A^b_{\mu c} A^c_{\nu a} - A^b_{\nu c} A^c_{\mu a} \; . \tag{A.77}$$

Suppressing the internal indices, we can write this as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$.

iv) Consequently, if D is a $\mathfrak{g}$-connection, also $F_{\mu\nu}$ is $\mathfrak{g}$-valued.

Theorem A.90 (Curvature Form)**.** *In local coordinates, we define the* $\operatorname{End}(E)$*-valued curvature 2-form to be $F = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. Hence, we can write the result of lemma A.89 as*

$$F = \mathrm{d}A + A \wedge A \; . \tag{A.78}$$

This is called Cartan's Structure Equation*.*
F is actually globally defined and coordinate-independent. (It is important to note though that (A.78) *only holds locally. If we write $D = D^0 + A$ globally, we will in general get $F = F^0 + \mathrm{d}A + A \wedge A$.)*

Proof. Applying d_D again to (A.74) gives, for an arbitrary E-valued form ω,

$$\mathrm{d}_D^2 \omega = F \wedge \omega \; . \tag{A.79}$$

This can be used to define F globally independent of coordinates. ☐

Theorem A.91 (Bianchi Identity)**.**

$$\mathrm{d}_D F = 0 \tag{A.80}$$

Proof. We can use lemma A.87 and definition (A.78) to calculate $\mathrm{d}_D F$. Some terms cancel and we get $\mathrm{d}_D F = \mathrm{d}^2 A + [A, A \wedge A]$, both terms are zero.
Alternatively, we could use definition (A.79) and compare $\mathrm{d}_D^3 \omega = \mathrm{d}_D(F \wedge \omega) = F \wedge \mathrm{d}_D \omega$. ☐

Theorem A.92. *Let D be a G-connection. Plugging $\tilde{A} = gAg^{-1} + g\mathrm{d}g^{-1}$ into* (A.78) *gives $\tilde{F} = gFg^{-1}$. This tells us the compatibility condition for F and how it transforms under gauge transformations.*

Definition A.93. The curvature of the Levi-Civita connection ∇ is called the *Riemann tensor* R:

$$R^\rho_{\sigma\,\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{A.81}$$

(note that all indices are of the same type and that the order of the bottom indices differs from (A.77)). The *Ricci tensor* is $\operatorname{Ric}_{\mu\nu} = R^\lambda_{\mu\,\lambda\nu}$ and the *curvature scalar* is $R = g^{\mu\nu} \operatorname{Ric}_{\mu\nu}$.

A.5.4 Application: Yang-Mills Theory

In physics, a *Yang-Mills gauge theory* describes fermionic matter which is interacting through bosonic fields. It is specified by giving a *gauge group*, the fermionic matter content and its representation under the gauge group.

Mathematically, Yang-Mills theory is just an application of the theory of vector bundles associated with a principal bundle. However, we will modify the definitions given above slightly to incorporate *coupling constants* and write everything in local coordinates:

- Let G be a Lie group and $\mathfrak{g}$ its Lie algebra with generators T^a $(a = 1, \ldots, \dim \mathfrak{g})$. The generators satisfy a commutation relation $[T^a, T^b] = \mathrm{i} f^{abc} T^c$.
- Choose a representation of G. The fermionic matter field ψ is classically a section of the vector bundle E associated with the principal G-bundle under the given representation (times an internal spinor structure).
- Locally, a gauge transformation is just an assignment of a group element $U(x)$ to every point x. We can write $U = \mathrm{e}^{-\mathrm{i}\alpha}$, where $\alpha(x) = \alpha^a(x) T^a$ is an element of $\mathfrak{g}$.

  Under a gauge transformation,
  $$\psi(x) \mapsto U(x)\psi(x) = \mathrm{e}^{-\mathrm{i}\alpha}\psi \tag{A.82}$$
  (equation (A.59)). Infinitesimally, that is $\psi \mapsto \psi - \mathrm{i}\alpha\psi$.
- A *connection* on the principal G-bundle induces a connection D on the associated vector bundle, acting on ψ. Like in (A.63), we write it in terms of a *vector potential*, but now we scale it with a coupling constant g:
  $$(D_\mu \psi)^i = \partial_\mu \psi^i + \mathrm{i} g A^i_{\mu j} \psi^j \ . \tag{A.83}$$
  The vector potential A is a $\mathfrak{g}$-valued 1-form. According to (A.70), it transforms as
  $$A_\mu \mapsto U A_\mu U^{-1} + \frac{\mathrm{i}}{g} (\partial_\mu U) U^{-1} \ . \tag{A.84}$$
  Infinitesimally: $A_\mu \mapsto A_\mu + \frac{1}{g} D_\mu \alpha$, where $D_\mu \alpha$ was defined in definition A.85.
- The kinetic term of ψ is
  $$\mathrm{i}\bar{\psi}\gamma^\mu D_\mu \psi \ , \tag{A.85}$$
  this is gauge invariant by lemma A.83.
- The curvature of the connection is (see lemma A.89)
  $$F_{\mu\nu} = \frac{1}{\mathrm{i}g}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu] \ . \tag{A.86}$$
  Under a gauge transformation, $F_{\mu\nu} \mapsto U F_{\mu\nu} U^{-1}$, this makes the kinetic term
  $$-\frac{1}{2}\operatorname{tr}(F \wedge *F) = -\frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{A.87}$$
  gauge invariant.

Note. Oftentimes, α is scaled with g as well and in all formulas α is replaced with $g\,\alpha$.

Appendix B

Complex Geometry

Note. This chapter was mainly taken from [58], [54, Ch. 8, 10.4 and 11], [48, Ch. 1.1] and [59, Ch. 1.1].

B.1 Complex Manifolds

B.1.1 Definitions

Definition B.1 (Complex Manifold)**.** First of all, a function $f : \mathbb{C}^n \to \mathbb{C}^m$ is *holomorphic* if each complex component is holomorphic (obeys the Cauchy-Riemann equations) in every complex variable.

Let X be a differentiable manifold. A *holomorphic atlas* contains charts $\varphi_i : U_i \to \mathbb{C}^m$ with holomorphic transition functions. X equipped with a holomorphic atlas is a *complex manifold*.

Two holomorphic atlases on X define the same *complex structure* on X if their union is a holomorphic atlas as well.

Definition B.2 (Complex Submanifold)**.** Let X be a complex manifold of $\mathbb{C}$-dimension m. A submanifold $Y \subset X$ is a *complex submanifold* of dimension $n \leq m$ if there is an atlas of charts $\varphi_i : U_i \to \mathbb{C}^m$ such that

$$\varphi_i(Y \cap U_i) \subset \mathbb{C}^n \ . \tag{B.1}$$

Example B.3 (The Torus T^2)**.** Take two points $\omega_1, \omega_2 \in \mathbb{C}$ with $\Im(\omega_2/\omega_1) > 0$. The *lattice* L defined by them is the set of all $n_1\omega_1 + n_2\omega_2$, $n_i \in \mathbb{Z}$. Such a lattice defines naturally a complex structure on the *torus* $T^2 = \mathbb{C}/L$.

One can show that two lattices define the same complex structure if and only if

$$\begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix} = \lambda M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{B.2}$$

for a $\lambda \in \mathbb{C}$ and a $M \in \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ (we factorize out the $\mathbb{Z}_2$ because a factor -1 in M can be put into λ) [54, Ex. 8.2].

Consequently, the complex structure is completely determined by the *modulus* $\tau = \omega_2/\omega_1 \in H = \{z \in \mathbb{C} : \Im(z) > 0\}$. Two τ result in the same complex structure if they are related by a $\mathrm{PSL}(2,\mathbb{Z})$ transformation. The moduli space of the torus is therefore $H/\mathrm{PSL}(2,\mathbb{Z})$. Details on how this space looks can be found e.g. in [60, Ch. 5.1].

Example B.4 (Complex Projective Space)**.** Another example is *complex projective space*, defined as $\mathbb{C}P^m = \left(\mathbb{C}^{m+1} - \{0\}\right) / \sim$ with $z \sim z' \Leftrightarrow z = \lambda z'$. The coordinate functions inherited from $\mathbb{C}^{m+1}$ are called *homogeneous coordinates*. The equivalence class $[(z^0, \ldots, z^m)]$ is usually written as $[z^0 : \cdots : z^m]$.

Every point has a neighborhood in which one of the homogeneous coordinates is nowhere zero. We can fix that coordinate to be 1, then the remaining m coordinates are unambigous. They are called *inhomogeneous coordinates* and define a chart on that neighborhood. $m+1$ of such charts cover $\mathbb{C}P^m$, take $U_\mu = \{[z^0 : \cdots : z^m] : z^\mu \neq 0\}$.

Since a complex manifold of $\mathbb{C}$-dimension m is also a differentiable manifold of dimension $2m$, we already know how to define (real) tangent and cotangent spaces. We write the m complex coordinate functions z_i as $2m$ real coordinate functions: $z^\mu = x^\mu + \mathrm{i}y^\mu$. T_pX is spanned by the $2m$ vectors $\{\partial_{x^\mu}, \partial_{y^\mu}\}$ and T_p^*X by the covectors $\{\mathrm{d}x^\mu, \mathrm{d}y^\mu\}$.

Definition B.5. The *complexified* tangent and cotangent spaces are

$$T_pX^{\mathbb{C}} = \mathbb{C} \otimes T_pX \qquad \text{and} \qquad T_p^*X^{\mathbb{C}} = \mathbb{C} \otimes T_p^*X \tag{B.3}$$

(note that $T_p^*X^{\mathbb{C}} = \left(T_pX^{\mathbb{C}}\right)^*$).

The vectors

$$\partial_{z^\mu} = \frac{1}{2}\left(\partial_{x^\mu} - \mathrm{i}\partial_{y^\mu}\right) \qquad \text{and} \qquad \partial_{\bar{z}^\mu} = \overline{\partial_{z^\mu}} = \frac{1}{2}\left(\partial_{x^\mu} + \mathrm{i}\partial_{y^\mu}\right) \tag{B.4}$$

are a basis of $T_pX^{\mathbb{C}}$. We also write $\partial_\mu = \partial_{z^\mu}$ and $\bar{\partial}_\mu = \partial_{\bar{z}^\mu}$. The corresponding dual basis of $T_p^*X^{\mathbb{C}}$ is

$$\mathrm{d}z^\mu = \mathrm{d}x^\mu + \mathrm{i}\,\mathrm{d}y^\mu \qquad \text{and} \qquad \mathrm{d}\bar{z}^\mu = \mathrm{d}x^\mu - \mathrm{i}\,\mathrm{d}y^\mu\ . \tag{B.5}$$

Definition B.6. The sheaf of *holomorphic functions* on X is called the *structure sheaf* $\mathcal{O}_X$. That means that for $U \subset X$ open,

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{C} : f \text{ holomorphic}\}\ . \tag{B.6}$$

Holomorphic functions $f : U \to \mathbb{C}$ or $\Phi : X \to Y$ on complex manifolds are defined analogously to smooth functions on a differentiable manifold.

We also define the sheaf $\mathcal{O}_X^*$ of nowhere vanishing holomorphic functions on X.

As a reminder:

Definition B.7 (Sheaf)**.** Let X be a topological space. A *sheaf* $\mathcal{F}$ gives us for every open $U \subset X$ an object $\mathcal{F}(U)$ of some category, and for $U \subset V$ there is a *restriction morphism* $r_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$. We write $r_{UV}(f) = f|_U$.

The morphisms have to satisfy the following axioms:

- $r_{UU} = \mathrm{id}$ and $r_{UV} \circ r_{VW} = r_{UW}$ (*pre-sheaf*)
- Let U_i be an open covering of U and $f, g \in \mathcal{F}(U)$. If $f|_{U_i} = g|_{U_i}$ for all i, then $f = g$.
- If we are given $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, there is $f \in \mathcal{F}(U)$ such that $f_i = f|_{U_i}$.

Note. The maximum principle has severe consequences on complex manifolds: For example, any global holomorphic function on a compact and connected X is constant, $\mathcal{O}_X(X) = \mathbb{C}$. Also, we cannot have a holomorphic partition of unity on a complex manifold (see definition A.44).

The definition of a meromorphic function is not so straightforward. We would like meromorphic functions on U to be the quotient field of $\mathcal{O}_X(U)$. This definition has a serious drawback: If U_i is a covering of X and f_i are meromorphic functions on U_i such that $f_i = f_j$ on $U_i \cap U_j$, we would expect it to extend to an f which is meromorphic on X. With the mentioned naive definition, this would not be the case however (see *Cousin problems*).

We need to define meromorphic functions as *local* quotients of holomorphic functions. The locality can be mathematically captured by considering stalks of the sheaf $\mathcal{O}_X$. We remember:

Definition B.8 (Stalk)**.** Let $\mathcal{F}$ be a sheaf on X and $x \in X$. The *stalk* of $\mathcal{F}$ at x is the direct limit $\mathcal{F}_x = \lim_{\overrightarrow{U \ni x}} \mathcal{F}(U)$. The direct limit $\lim_\to$ is defined as the disjoint union of all elements of all $\mathcal{F}(U)$, where two elements $f \in \mathcal{F}(U), g \in \mathcal{F}(V)$ are identified if there is a $W \subset U \cap V$ containing x such that $f|_W = g|_W$.

Elements of a stalk are called *germs*.

Definition B.9 (Meromorphic Function)**.** A *meromorphic function* f associates to every $p \in X$ a germ $f_p \in Q(\mathcal{O}_{X,p})$ (Q denotes the quotient field). This happens in such a way that for every point of X there is a neighborhood U and $g, h \in \mathcal{O}_X(U)$ with $f_p = \frac{[g]}{[h]}$ for all $p \in U$ (where $[g]$ and $[h]$ are the equivalence classes of g and h in the stalk $\mathcal{O}_{X,p}$).

$\mathcal{K}_X$ is the sheaf of meromorphic functions and $\mathcal{K}_X^*$ the sheaf of invertible meromorphic functions.

B.1.2 Almost Complex Structure

We continue to investigate the tangent space of a complex manifold. In every point $p \in X$ we define J_p, a $(1,1)$-tensor of T_pX, by setting $J_p \partial_{x^\mu} = \partial_{y^\mu}$ and $J_p \partial_{y^\mu} = -\partial_{x^\mu}$. Using that the transition functions between charts are holomorphic and thus obey the Cauchy-Riemann equations it can be easily seen that this definition is independent of the chart and gives us a $(1,1)$-tensor field J.

Definition B.10 (Almost Complex Structure)**.** An *almost complex structure* on a differentiable manifold is a $(1,1)$-tensor field J that squares to

$$J^2 = -\,\mathrm{id} \ . \tag{B.7}$$

An almost complex structure always has m eigenvalues $+\mathrm{i}$ and m eigenvalues $-\mathrm{i}$. The complexified tangent space splits into two subspaces:

$$T_pX^{\mathbb{C}} = T_pX^+ \oplus T_pX^- \ . \tag{B.8}$$

Note. In a complex manifold, T_pX^+ is spanned by the ∂_μ and T_pX^- by the $\bar{\partial}_\mu$.

Definition B.11. Vectors in T_pX^+ (T_pX^-) are called *(anti-)holomorphic vectors*.
Also vector fields can be decomposed accordingly and vector fields in $\mathrm{Vect}(X)^+$ ($\mathrm{Vect}(X)^+$) are called *(anti-)holomorphic vector fields*.

Theorem B.12 (Newlander and Nirenberg 1957)**.** *Let M be a $2m$-dimensional differentiable manifold with almost complex structure J. If $[v,w] \in \mathrm{Vect}(X)^+$ for all $v,w \in \mathrm{Vect}(X)^+$, we say that J is* integrable.

If J is integrable, then M is a complex manifold with almost complex structure J. On the other hand, the almost complex structure of every complex manifold is integrable.

Proof. [58, Thm. 2.6.19] □

B.1.3 Complex Differential Forms

Definition B.13 ((r,s)-Form)**.** A q-vector $\omega \in \Omega^q_p(X)^{\mathbb{C}}$ is said to be an *(r,s)-vector* or vector of *bidegree* (r,s) if $\omega(V_1,\dots,V_q) = 0$ unless r of the vectors are holomorphic and s of the vectors are antiholomorphic. The set of (r,s)-forms is called $\Omega^{r,s}(X)$.

A generic form ω of bidegree (r,s) can locally be written as

$$\omega = \frac{1}{r!s!}\omega_{\mu_1\dots\mu_r\nu_1\dots\nu_s}\, \mathrm{d}z^{\mu_1} \wedge \dots \wedge \mathrm{d}z^{\mu_r} \wedge \mathrm{d}\bar{z}^{\nu_1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\nu_s} \ . \tag{B.9}$$

Lemma B.14 (Decomposition)**.**

$$\Omega^q(X)^{\mathbb{C}} = \bigoplus_{r+s=q} \Omega^{r,s}(X) \tag{B.10}$$

In this context, the wedge product is a map $\wedge : \Omega^{r,s} \times \Omega^{r',s'} \to \Omega^{r+r',s+s'}$. If we apply the exterior derivative to an (r,s)-form, we get a form in $\Omega^{r+1,s} \oplus \Omega^{r,s+1}$.

Definition B.15 (Dolbeault Operators)**.** The *Dolbeault operators* ∂ and $\bar{\partial}$ are the induced maps $\partial : \Omega^{r,s} \to \Omega^{r+1,s}$ and $\bar{\partial} : \Omega^{r,s} \to \Omega^{r,s+1}$ such that $\mathrm{d} = \partial + \bar{\partial}$. For $\omega = \frac{1}{r!s!}\omega_{\boldsymbol{\mu\nu}}\, \mathrm{d}z^{\boldsymbol{\mu}} \wedge \mathrm{d}\bar{z}^{\boldsymbol{\nu}}$ (short notation for (B.9)) they are given by

$$\partial\omega = \frac{1}{r!s!}\partial_\lambda \omega_{\boldsymbol{\mu\nu}}\, \mathrm{d}z^\lambda \wedge \mathrm{d}z^{\boldsymbol{\mu}} \wedge \mathrm{d}\bar{z}^{\boldsymbol{\nu}} \tag{B.11}$$

$$\bar{\partial}\omega = \frac{1}{r!s!}\bar{\partial}_\lambda \omega_{\boldsymbol{\mu\nu}}\, \mathrm{d}\bar{z}^\lambda \wedge \mathrm{d}z^{\boldsymbol{\mu}} \wedge \mathrm{d}\bar{z}^{\boldsymbol{\nu}} \ . \tag{B.12}$$

A q-form ω of bidegree $(q,0)$ with $\bar{\partial}\omega = 0$ is called *holomorphic*.

Lemma B.16. *The usual rules for computing hold: $\partial^2 = 0 = \bar{\partial}^2$, and*

$$\partial(\omega \wedge \eta) = \partial\omega \wedge \eta + (-1)^q \omega \wedge \partial\eta$$

($\bar{\partial}$ analogously).

Definition B.17 (Dolbeault Complex)**.** The cochain complex

$$0 \to \Omega^{r,0}(X) \xrightarrow{\bar{\partial}} \Omega^{r,1}(X) \xrightarrow{\bar{\partial}} \cdots \tag{B.13}$$

is called the *Dolbeault complex* of X.

Its cohomology groups are the (r,s) *$\bar{\partial}$-cohomology groups* $H^{r,s}_{\bar{\partial}}(X)$. The (complex) dimensions of those are the *Hodge numbers* $b^{r,s}$.

B.2 Kähler Manifolds

B.2.1 Hermitian Manifolds

Definition B.18 (Hermitian Manifold)**.** Let g be a Riemannian metric on a complex manifold X. If

$$g_p(J_pY, J_pZ) = g_p(Y,Z) \tag{B.14}$$

for all $Y, Z \in T_pX$ then g is a *Hermitian metric* and (X,g) a *Hermitian manifold.*

Theorem B.19. *Every complex manifold can be made into a hermitian manifold.*

Proof. Take some Riemannian metric $\hat{g}$ and set $g(v,w) = \frac{1}{2}(\hat{g}(v,w) + \hat{g}(Jv,Jw))$. □

Definition B.20 (Complex Extension)**.** g_p can be extended to a $\mathbb{C}$-bilinear map $g_p : T_pX^{\mathbb{C}} \otimes T_pX^{\mathbb{C}} \to \mathbb{C}$. The extension is still just called g. Now we can give the components in the basis $\{\mathrm{d}z^\mu, \mathrm{d}\bar{z}^\nu\}$:

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) \qquad \text{and} \qquad g_{\mu\bar{\nu}} = g(\partial_\mu, \bar{\partial}_\nu) \qquad \text{etc .} \tag{B.15}$$

Lemma B.21. *Because g is bilinear, $g_{\mu\nu} = g_{\nu\mu}$ and $g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu}$ etc.*
Because g originally was real, $\overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}}$ and $\overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu}$ etc.
Because g is hermitian, one easily sees $g_{\mu\nu} = 0 = g_{\bar{\mu}\bar{\nu}}$. Therefore

$$g = g_{\mu\bar{\nu}} \mathrm{d}z^\mu \otimes \mathrm{d}\bar{z}^\nu + g_{\bar{\mu}\nu} \mathrm{d}\bar{z}^\mu \otimes \mathrm{d}z^\nu \ . \tag{B.16}$$

Definition B.22 (Kähler Form)**.** Let (M,g) be a Hermitian manifold. We define $\Omega(v,w) = g(Jv,w)$.

Ω is a real two-form ($\Omega \in \Omega^2(X)$) called *fundamental form* or *Kähler form.* After extension to $\mathbb{C}$, its components are $\Omega_{\mu\bar{\nu}} = -\Omega_{\bar{\nu}\mu} = \mathrm{i}g_{\mu\bar{\nu}}$, so

$$\Omega = \mathrm{i}g_{\mu\bar{\nu}}\, \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu \ . \tag{B.17}$$

Proof. Ω is real by definition. Its antisymmetry can be seen from $\Omega(v,w) = g(J^2v, Jw) = -g(Jw,v) = -\Omega(w,v)$. □

Theorem B.23. *A Hermitian manifold is orientable.*

Proof. $\Omega \wedge \cdots \wedge \Omega$ is a nowhere vanishing $2m$-form. This can be easily seen by applying it to a basis of the form $\{\boldsymbol{e}_1, J\boldsymbol{e}_1, \dots, \boldsymbol{e}_m, J\boldsymbol{e}_m\}$ in any point. □

Note. Some authors call $h = \frac{1}{2}(g - \mathrm{i}\Omega)$ the hermitian metric. The Riemannian metric g is then defined to be $2\Re(h)$ and Ω is $-2\Im(h)$. The only nonzero components of h are $h_{\mu\bar{\nu}} = g_{\mu\bar{\nu}}$:

$$h = g_{\mu\bar{\nu}}\, \mathrm{d}z^\mu \otimes \mathrm{d}\bar{z}^\nu \ . \tag{B.18}$$

h_p can be seen as a map $h_p : T_pX^+ \otimes T_pX^- \to \mathbb{C}$ which actually is hermitian, meaning $h_p(Y,\bar{Z}) = \overline{h_p(Z,\bar{Y})}$ for $Y, Z \in T_pX^+$.

On the tangent bundle of a Hermitian manifold (X, g), we can define the *Levi-Civita connection* ∇ just like in definition A.80. On the complexified tangent bundle $TX^{\mathbb{C}}$, there is another natural connection:

Definition B.24 (Chern Connection)**.** Let D be a real connection on $TX^{\mathbb{C}}$ with vector potential A. We extend the argument v of A_v to take complex vector fields by simple $\mathbb{C}$-linear extension. Now, D is *compatible with the complex structure* if $A_\mu(TX^+) \subset TX^+$ (i.e. $A^{\bar{\lambda}}_{\mu *} = 0$)) and $A_{\bar{\mu}}(TX^+) = 0$ (i.e. $A^*_{\mu\bar{\nu}} = 0$). Because D is real, we directly see that this is equivalent to all mixed-index components of A being zero.

The *Chern connection* D is the unique real connection which is compatible with the metric (like in definition A.80) and the complex structure. Its vector potential is called Γ (like the vector potential of ∇).

Lemma B.25. *Because* D *is compatible with the complex structure, the only non-zero components of* Γ *are* $\Gamma^\lambda_{\mu\nu} = \overline{\Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}}}$. *They can be calculated via*

$$\Gamma^\lambda_{\mu\nu} = g^{\lambda\bar{\kappa}} \partial_\mu g_{\nu\bar{\kappa}} \qquad \text{and} \qquad \Gamma^{\bar{\lambda}}_{\bar{\mu}\bar{\nu}} = g^{\bar{\lambda}\kappa} \bar{\partial}_\mu g_{\bar{\nu}\kappa} \ . \tag{B.19}$$

This also shows the uniqueness of the Chern connection.

Theorem B.26. *The almost complex structure J is* parallel *with respect to the Chern connection* D*:*

$$\mathrm{D}J = 0 \ . \tag{B.20}$$

B.2.2 Kähler Manifolds

Definition B.27 (Kähler Manifold)**.** A *Kähler manifold* is a Hermitian manifold with closed Kähler form, $\mathrm{d}\Omega = 0$.

Theorem B.28. *A Hermitian manifold is Kähler if and only if J is also parallel with respect to the* Levi-Civita connection*,*

$$\nabla J = 0 \ . \tag{B.21}$$

Proof. First one notes that for any form ω, $\mathrm{d}\omega = \nabla\omega$ because of torsion-freeness.

From calculating $(\nabla_v \Omega)(u, w)$ one then finds out that ∇J is zero if and only if $\nabla\Omega$ is zero. This proves the claim because $\mathrm{d}\Omega = \nabla\Omega$. □

Theorem B.29. *The condition* $\mathrm{d}\Omega = 0$ *can be locally written as*

$$\partial_\lambda g_{\mu\bar{\nu}} = \partial_\mu g_{\lambda\bar{\nu}} \qquad \text{and} \qquad \partial_{\bar{\lambda}} g_{\bar{\mu}\nu} = \partial_{\bar{\mu}} g_{\bar{\lambda}\nu} \ . \tag{B.22}$$

This has the following consequences:

i) The Chern connection is torsion-free and therefore agrees with the (complexified) Levi-Civita connection.

ii) In a chart U_i, $g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \mathcal{K}_i$ for some Kähler potential $\mathcal{K}_i$*. In other words, $\Omega = \mathrm{i}\partial\bar{\partial}\mathcal{K}_i$.*

Proof. To get (B.22) just calculate $(\partial + \bar{\partial})\,(\mathrm{i} g_{\mu\bar{\nu}}\, \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\nu)$ which has to be zero. □

Example B.30. The Euclidean metric δ on $\mathbb{C}^m$ is Hermitian. Its Kähler form $\Omega = \frac{\mathrm{i}}{2} \sum_\mu \mathrm{d}z^\mu \wedge \mathrm{d}\bar{z}^\mu$ is obviously closed, it has a Kähler potential $\mathcal{K} = \frac{1}{2} z_\mu \bar{z}^\mu$.

Example B.31. Complex projective space is a Kähler manifold. Define a Kähler potential by setting (using the charts of example B.4)

$$\mathcal{K}_\mu([z^0 : \cdots : z^m]) = \frac{1}{2\pi} \sum_{\nu=0}^{m} \left| \frac{z^\nu}{z^\mu} \right|^2 \tag{B.23}$$

This defines a Kähler form called $\omega_{FS} = \mathrm{i}\partial\bar{\partial}\mathcal{K}$ because $\partial\bar{\partial}K_\mu = \partial\bar{\partial}K_\nu$. There is a Hermitian metric g whose Kähler form is Ω, it is called the *Fubini-Study metric.*

Definition B.32 (Projective Manifold)**.** A complex manifold X is *projective* if it is isomorphic to a closed complex submanifold of a complex projective space.

Theorem B.33. *Every projective manifold is Kähler.*

Proof. It is easy to show that, in general, a complex submanifold of a Kähler manifold is again Kähler. □

Lemma B.34 (Curvature of Hermitian Manifolds)**.** *Using lemma B.25, we see the following:*

Let $R^{\rho}_{\sigma\,\mu\bar{\nu}}$ be the curvature tensor corresponding to the Chern connection on a hermitian manifold, defined like in A.93. In general the only non-vanishing components of it are:

$$R^{\rho}_{\sigma\,\mu\bar{\nu}} = -R^{\rho}_{\sigma\,\bar{\nu}\mu} \tag{B.24}$$

and their complex conjugates.

Theorem B.35 (Curvature of Kähler Manifolds)**.** *On a Kähler manifold, the Levi-Civita and the Chern connection are the same.* (B.24) *is therefore true for the Riemann tensor $R^{\rho}_{\sigma\,\mu\bar{\nu}}$.*

(B.22) *implies an additional symmetry of the Riemann tensor:*

$$R^{\rho}_{\sigma\,\mu\bar{\nu}} = R^{\rho}_{\mu\,\sigma\bar{\nu}} \; . \tag{B.25}$$

Importantly,

$$\mathrm{Ric}_{\mu\bar{\nu}} = R^{\lambda}_{\mu\,\lambda\bar{\nu}} = R^{\lambda}_{\lambda\,\mu\bar{\nu}} = \mathrm{tr}(R_{\mu\bar{\nu}}) \tag{B.26}$$

where the trace goes over the "endomorphism"-part of R viewed as an endomorphism-valued 2*-form. We will need this later for the characterization of Calabi-Yau manifolds (lemma B.92).*

Note that on a hermitian manifold, $\mathrm{tr}(R_{\mu\bar{\nu}}) = -\partial_\mu\bar{\partial}_\nu \log\det g$ *(from lemma B.25).*

B.2.3 Hodge Theory on Hermitian Manifolds

On a Hermitian manifold, we can define the Hodge star operator like in definition A.33. It is a map $* : \Omega^{r,s}(X) \to \Omega^{m-s,m-r}(X)$. Then we define a scalar product on $\Omega^{r,s}(X)$ similar to definition A.51:

$$(\omega, \eta) = \int_M \omega \wedge \overline{*\eta} \; . \tag{B.27}$$

The *adjoint Dolbeault operators* $\partial^\dagger$ and $\bar{\partial}^\dagger$ are defined with respect to this scalar product similar to definition A.52. It turns out that $\mathrm{d}^\dagger = \partial^\dagger + \bar{\partial}^\dagger$, $\partial^\dagger = -*\bar{\partial}*$ and $\bar{\partial}^\dagger = -*\partial*$, also $(\partial^\dagger)^2 = 0 = (\bar{\partial}^\dagger)^2$.

Definition B.36 (Harmonic Forms)**.** On a Hermitian manifold, we define $\Delta_\partial = (\partial + \partial^\dagger)^2$ and $\Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^\dagger)^2$.

We call the space of *harmonic forms* with respect to $\Delta_{\bar{\partial}}$

$$\mathrm{Harm}^{r,s}_{\bar{\partial}}(X) = \{\omega \in \Omega^{r,s}(X) : \Delta_{\bar{\partial}}\omega = 0\} \; . \tag{B.28}$$

In this context, we have analogues of theorems A.58 and A.59:

Theorem B.37 (Hodge)**.** *There is the orthogonal decomposition*

$$\Omega^{r,s}(X) = \bar{\partial}\Omega^{r,s-1}(X) \oplus \bar{\partial}^\dagger\Omega^{r,s+1}(X) \oplus \mathrm{Harm}^{r,s}_{\bar{\partial}}(X) \; . \tag{B.29}$$

Also, the $\bar{\partial}$-homology groups (see definition B.17) are isomorphic to spaces of harmonic forms:

$$H^{r,s}_{\bar{\partial}}(X) \cong \mathrm{Harm}^{r,s}_{\bar{\partial}}(X) \; . \tag{B.30}$$

B.2.4 Cohomology of Kähler Manifolds

Theorem B.38. *Let X be a Kähler manifold. Then*

$$\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial} \ . \tag{B.31}$$

Definition B.39 (Hodge Diamond)**.** The Hodge numbers defined in definition B.17 are usually arranged in the *Hodge diamond*:

$$\begin{pmatrix}
 & & & & b^{m,m} & & & & \\
 & & & b^{m,m-1} & & b^{m-1,m} & & & \\
 & & \iddots & & & & \ddots & & \\
 & b^{m,1} & & & \vdots & & & b^{1,m} & \\
b^{m,0} & & & \cdots & \cdots & \cdots & & & b^{0,m} \\
 & b^{m-1,0} & & & \vdots & & & b^{0,m-1} & \\
 & & \ddots & & & & \iddots & & \\
 & & & b^{1,0} & & b^{0,1} & & & \\
 & & & & b^{0,0} & & & &
\end{pmatrix} \tag{B.32}$$

Theorem B.40. *In a Kähler manifold,*

i) the Hodge Diamond is symmetric about the vertical axis: $b^{r,s} = b^{s,r}$.

ii) the Hodge Diamond is symmetric about the horizontal axis: $b^{r,s} = b^{m-s,m-r}$.

Proof. i) If $\omega \in \Omega^{r,s}$ is $\bar\partial$-harmonic then $\bar\omega \in \Omega^{s,r}$ is also $\bar\partial$-harmonic:

$$\Delta_{\bar\partial}\bar\omega = \overline{\Delta_\partial \omega} = \overline{\Delta_{\bar\partial}\omega} = 0 \ .$$

ii) The map Λ which we already used in equation (A.41) can be seen as a bilinear non-degenerate map $H^{r,s} \times H^{m-r,m-s} \to \mathbb{C}$. This gives $b^{r,s} = b^{m-r,m-s} = b^{m-s,m-r}$. □

B.3 Holomorphic Vector Bundles

B.3.1 Interlude: Cohomology of Sheaves

There is a very general notion of cohomology groups $H^q(X, \mathcal{F})$ of sheaves $\mathcal{F}$. The definition that works most universally is *Grothendieck cohomology* (usually just called "cohomology"): Here the homology group functor $H^q(X, \cdot)$ is defined as the right derived functor of the section functor $\Gamma_X : \mathcal{F} \to \mathcal{F}(X)$. *Čech cohomology* $\check{H}(X, \mathcal{F})$ is a less abstract approach which often agrees with Grothendieck cohomology (see theorem B.43 and [58, 61]), but is more easily accessible and calculable.

Definition B.41 (Čech cohomology)**.** To define Čech cohomology, we first fix an open cover $\{U_i : i \in I\}$ of X.

The *cochain groups* are defined as follows:

$$C^n(\{U_i\}, \mathcal{F}) = \prod_{i_0 \neq \cdots \neq i_n} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_n}) \ . \tag{B.33}$$

This means that a 0-cochain $\sigma \in C^0$ is a collection of objects $\sigma_i \in F(U_i)$. 1-cochains are collections of the type $\sigma_{ij} \in F(U_i \cap U_j)$ and so on.

We define the *coboundary* of a cochain in a way similar to (A.33). For example, for a 0-cochain σ, the coboundary is a 1-cochain $\delta\sigma$ with $(\delta\sigma)_{ij} = -\left.(\sigma_i)\right|_{U_i \cap U_j} + \left.(\sigma_j)\right|_{U_i \cap U_j}$. In general, for an n-cochain σ,

$$(\delta\sigma)_{i_0 \ldots i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j \left. \left(\sigma_{i_0 \ldots \hat{i}_j \ldots i_{n+1}}\right)\right|_{U_{i_0} \cap \cdots \cap U_{i_{n+1}}} \ . \tag{B.34}$$

Since $\delta^2 = 0$, the cochain groups together with the coboundary operators make up a cochain complex. Let's call its cohomology groups $H^q(\{U_i\}, \mathcal{F})$.

Čech cohomology is the direct limit of the groups $H^q(\{U_i\}, \mathcal{F})$ for ever finer coverings $\{U_i\}$:

$$\check{H}^q(X, \mathcal{F}) = \varinjlim_{\{U_i\}} H^q(\{U_i\}, \mathcal{F}) \ . \tag{B.35}$$

Taking the direct limit is possible because if $\{V_i\}$ is a finer covering then $\{U_i\}$ then there is a natural homomorphism $H^q(\{U_i\}, \mathcal{F}) \to H^q(\{V_i\}, \mathcal{F})$.

Lemma B.42. *For every $\mathcal{F}$, there is an open cover $\{U_i\}$ such that $\check{H}^q(X, \mathcal{F}) = H^q(\{U_i\}, \mathcal{F})$ [61].*

Theorem B.43. *There is a natural isomorphism between the zeroth cohomology groups, $H^0(X, \mathcal{F}) \cong \check{H}^0(X, \mathcal{F})$, and between the first cohomology groups, $H^1(X, \mathcal{F}) \cong \check{H}^1(X, \mathcal{F})$.*

The higher cohomology groups agree if for example X is paracompact.

For the proof of this theorem as well as for the following ones we would need to know how to compute Grothendieck cohomology using an acyclic resolution of the sheaf $\mathcal{F}$. Details about this are in [58].

Note. Obviously, the zeroth homology group just contains the global sections of the sheaf,

$$H^0(X, \mathcal{F}) = \mathcal{F}(X) \ . \tag{B.36}$$

Theorem B.44. *Let M be a differentiable manifold.*

We denote the constant sheaf with values in $\mathbb{R}$ by $\underline{\mathbb{R}}$.

$$H^q(M, \underline{\mathbb{R}}) \cong H^q(M) \cong H_q^*(M) \ . \tag{B.37}$$

Here $H^q(M)$ is the deRham cohomology (definition A.46). We have already seen in theorem A.48 that it is the dual of the singular homology (with real coefficients) $H_q(M)$, defined in A.41.

Theorem B.45. *Let X be a complex manifold.*

The holomorphic r-forms on it form a sheaf $\Omega^{r,0}(X) = \Omega_X^r$ (see definitions B.13, B.51 and B.52).

$$H^s(X, \Omega_X^r) \cong H_{\bar\partial}^{r,s}(X) \ , \tag{B.38}$$

where $H_{\bar\partial}$ is the Dolbeault cohomology already defined in definition B.17.

This also works for E-valued forms if $\pi : E \to X$ is a holomorphic vector bundle (to be defined below):

$$H^s(X, E \otimes \Omega_X^r) \cong H_{\bar\partial}^{r,s}(X, E) \ , \tag{B.39}$$

where $H_{\bar\partial}^{r,s}(X, E)$ are the cohomology groups of the E-valued Dolbeault complex.

Note. One can understand why all those different types of cohomology usually agree: It is possible to define (co-)homology axiomatically, the axioms are called *Eilenberg-Steenrod axioms*. Those axioms were all already mentioned in subsection A.3.2 as properties of singular homology: The dimension axiom, additivity, excision, the homotopy axiom and the existence of the long exact sequence in homology.

The *Eilenberg-Steenrod uniqueness theorem* (see e.g. [62]) tells us that if the underlying manifold is e.g. a finite *CW-complex* [57, Ch. 4], (co-)homology is unique. Since all the definitions of (co-)homology we have seen satisfy the axioms, they have to agree.

B.3.2 Holomorphic Vector Bundles

Definition B.46 (Complex Vector Bundle)**.** Let E and X be complex manifolds. A vector bundle $F \to E \xrightarrow{\pi} X$ is called a *complex vector bundle* of rank r if the fiber F is $\mathbb{C}^r$.

Definition B.47 (Holomorphic Vector Bundle)**.** A complex vector bundle $\mathbb{C}^r \to E \xrightarrow{\pi} X$ with structure group $\mathrm{GL}(\mathbb{C}, r)$ is a *holomorphic vector bundle* if

i) π is holomorphic as a map between complex manifolds.

ii) The transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}(\mathbb{C}, r)$ are holomorphic.

iii) The local trivializations $\varphi_i : U_i \times \mathbb{C}^r \to E|_{U_i}$ are biholomorphic.

A rank 1 holomorphic vector bundle is called a *holomorphic line bundle*.

Theorem B.48 (Cocycle Description)**.** *Theorem A.65 still holds true: A holomorphic vector bundle is uniquely specified by giving the base space, the rank, and charts and transition functions satisfying the cocycle conditions.*

Example B.49. The *trivial holomorphic line bundle* is $\mathcal{O} = X \times \mathbb{C}$. Its sheaf of sections is the structure sheaf $\mathcal{O}_X$ of holomorphic functions.

A holomorphic line bundle is trivial if and only if it admits a global section that is nowhere vanishing and holomorphic.

Example B.50. The *holomorphic tangent bundle* TX^+ (see definition B.11) is a holomorphic vector bundle. Its rank is equal to the (complex) dimension of the complex manifold X.

Proof. Let $\zeta = \zeta(z)$ be a change of chart. By definition it is holomorphic, so $\partial_{z^\mu} = \frac{\partial \zeta^\nu}{\partial z^\mu} \partial_{\zeta^\nu}$. The components of a vector $v_p = v_p^\mu \left(\partial_{z^\mu}\right)_p \in T_p X^+$ then transform with the holomorphic matrix $\frac{\partial \zeta^\nu}{\partial z^\mu}$. ☐

Example B.51. The *holomorphic cotangent bundle* $\Omega^{1,0}(X)$ is by definition the dual bundle of TX^+. For convenience we will call it $\mathit{\Omega}_X$.

By taking the q-fold exterior product, we get the bundle of *holomorphic q-forms* $\Lambda^q \mathit{\Omega}_X = \Omega^{q,0}(X)$ which we will call $\mathit{\Omega}_X^q$. Note that on an m-dimensional complex manifold X there are q-forms for $q \leq 2m$, but only holomorphic q-forms for $q \leq m$.

The *canonical bundle* K_X is the bundle of holomorphic m-forms:

$$K_X = \mathit{\Omega}_X^m = \Omega^{m,0}(X) = \det \mathit{\Omega}_X \ . \tag{B.40}$$

(The *determinant bundle* of a rank r bundle E is $\det E = \Lambda^r E$.)

Definition B.52 (Sheaf of Sections)**.** Let $\pi : E \to X$ be a holomorphic vector bundle. The holomorphic sections of E form a sheaf over X, we will call this sheaf E as well. This should clarify the notation in theorem B.45.

Theorem B.53 (Adjunction Formula)**.** *Let $Y \subset X$ be a complex submanifold according to definition B.2. Then naturally $TY^+ \subset T(X|_Y)^+$. The orthogonal complement of TY^+ is again a holomorphic vector bundle, called the* normal bundle $\mathcal{N}_{Y/X}$.

There is a natural isomorphism

$$K_Y \cong K_{X|_Y} \otimes \det \mathcal{N}_{Y/X} \ . \tag{B.41}$$

Proof. It is true in general that for a short exact sequence $0 \to E \to F \to G \to 0$ of holomorphic vector bundles, $\det F \cong \det E \otimes \det G$. Applying this to

$$0 \to TY^+ \to T(X|_Y)^+ \to \mathcal{N}_{Y/X} \to 0 \tag{B.42}$$

yields the claim.

We can also see this by straightforward calculation: If Y is a submanifold of X this means that the transition functions g_{ij}^X of $X|_Y$ can be chosen block-diagonal: One block consists of the transition functions g_{ij}^Y of Y and one of those of $\mathcal{N}_{Y/X}$, $g_{ij}^{\mathcal{N}}$. The transition functions of the canonical bundle are the determinant of the inverse transpose matrix (see the definition A.69 of the dual bundle). The theorem is true because

$$\left(\det g_{ij}^Y\right)^{-1} = \left(\det g_{ij}^X\right)^{-1} \left(\det g_{ij}^{\mathcal{N}}\right)$$

and the transition functions determine the bundle according to theorem B.48. ☐

Definition B.54 (Picard Group)**.** The *Picard group* $\mathrm{Pic}(X)$ of a complex manifold X consists of (isomorphism classes of) holomorphic line bundles over X. The group operation is taking the tensor product, the inverse is given by taking the dual bundle and the trivial line bundle $\mathcal{O}$ is the neutral group element.

Proof. The only nontrivial thing we have to show is that for a line bundle $\mathbb{C} \to L \xrightarrow{\pi} X$, $L \otimes L^* \cong \mathcal{O}$. The easiest way to see that is using theorem B.48: Let g_{ij} be the transition functions of L, then the transition functions of $L \otimes L^*$ are $g_{ij} \otimes (g_{ij})^{-T} = 1$ (because g_{ij} are 1×1 matrices). □

Theorem B.55. *The Picard group* $\mathrm{Pic}(X)$ *is naturally isomorphic to* $H^1(X, \mathcal{O}_X^*)$*.*
($\mathcal{O}_X^*$ is the sheaf of nowhere vanishing holomorphic functions like in definition B.6.)

Proof. First of all, by theorem B.43 we can use $\check{H}^1$ instead of H^1. According to lemma B.42, we can fix a cover $\{U_i\}$ fine enough to compute Čech cohomology with it.

Let us now map a line bundle $L \in \mathrm{Pic}(X)$ to the 1-cochain $(g)_{ij} = g_{ij}$ of its transition functions. This cochain is closed because

$$(\delta g)_{ijk} = g_{jk} g_{ik}^{-1} g_{ij} = e$$

due to the cocycle condition of the transition functions.

Therefore we have defined a homomorphism $\mathrm{Pic}(X) \to \check{H}^1(X, \mathcal{O}_X^*)$. The homeomorphism is surjective because the transition functions determine the line bundle (theorem B.48). It is also injective because its kernel consists of line bundles with transition functions of the form $g_{ij} = h_j h_i^{-1}$ for some h_i which are holomorphic and nowhere vanishing – such a bundle is trivial (see example B.49). □

B.3.3 Divisors of Complex Manifolds

Definition B.56 (Analytic Subvariety)**.** Let X be a complex manifold. A closed subset $Y \subset X$ is an *analytic subvariety* if for each $x \in X$ there is an open neighborhood $U_x \subset X$ such that $Y \cap U_x$ can be described as the zero set of a finite set of holomorphic functions over U_x.

A point $p \in Y$ is *regular* if there is an open neighborhood V_p of p such that $Y \cap V_p$ is a complex manifold. Otherwise, p is called *singular*. The *dimension* of Y in a regular point p is $\dim_p Y = \dim(Y \cap V_p)$. The dimension of Y is $\dim Y = \sup\{\dim_p Y : p \text{ regular}\}$.[1]

An analytic subvariety Y is an (analytic) *hypersurface* if $\dim Y = \dim X - 1$.

Definition B.57 (Irreducible Subvariety)**.** An analytic subvariety Y is called *irreducible* if it can not be written as the union of two proper analytic subvarieties.

Definition B.58 (Divisor)**.** The group of *divisors* $\mathrm{Div}(X)$ of a complex manifold X is the free abelian group over the set of irreducible hypersurfaces. In other words, elements $D \in \mathrm{Div}(X)$ are of the form

$$D = \sum_i a_i Y_i \tag{B.43}$$

for $a_i \in \mathbb{Z}$ and irreducible hypersurfaces Y_i.

A divisor is called *effective* if all the a_i are not negative.

The next step is to define divisors corresponding to meromorphic functions. For this, we need the concept of the order of a meromorphic function along an irreducible hypersurface:

Definition B.59 (Order)**.** Let $f \in \mathcal{K}_X(X)$ be a globally defined meromorphic function and Y an irreducible hypersurface in X.

Fix a point $x \in Y$. According to the definition, Y is given as the zero set of a holomorphic function $g \in \mathcal{O}_{X,x}$ in a neighborhood of x. Because $\mathcal{O}_{X,x}$ is a unique factorization domain [58, Prop. 1.1.15] and g is irreducible, we can write

$$f_x = g^n \cdot \ldots \tag{B.44}$$

with a unique exponent n, this is the *order of f along Y in x*: $\mathrm{ord}_{Y,x}(f) = n$.

[1] Alternatively we could use that the subset of regular points Y_{reg} is a complex submanifold and set $\dim Y = \dim Y_{\mathrm{reg}}$.

The order in x is actually the same for all regular points $x \in Y$ since it doesn't change in a small neighborhood and the set of regular points of Y is connected. This allows us to define

$$\mathrm{ord}_Y(f) = \mathrm{ord}_{Y,x}(f), \text{ for } x \in Y \text{ regular.} \tag{B.45}$$

Definition B.60 (Principal Divisor)**.** Now we can assign a divisor

$$(f) = \sum_{Y \text{ irr.}} \mathrm{ord}_Y(f)\, Y \tag{B.46}$$

to each $f \in \mathcal{K}_X(X)$. Divisors of this form are called *principal divisors.*

Lemma B.61. *What we have defined so far are* Weil divisors*. In contrast,* Cartier divisors *are defined to be elements of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ (taking the quotient is possible because a nowhere vanishing holomorphic function is also an invertible meromorphic function).*

In our setting, those definitions agree:

$$\mathrm{Div}(X) \cong H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)\ . \tag{B.47}$$

Proof. An element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is first of all a collection of $f_i \in \mathcal{K}_X^*(U_i)$. Two such collections (f_i) and (g_i) are equivalent if $f_i = g_i \cdot h_i$ for a $h_i \in \mathcal{O}_X^*(U_i)$ for all i. Also (f_i) has to be coclosed, i.e. $(\delta f)_{ij} = f_i^{-1} \cdot f_j \in \mathcal{O}_X^*(U_i \cap U_j)$.

Because of this last condition, the notion $\mathrm{ord}_Y(f)$ is well-defined for an $f \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ and we assign to it the divisor $(f) = \sum_{Y \text{ irr.}} \mathrm{ord}_Y(f)\, Y$. (Note: This formula is different to (B.46) because f is a different kind of object!) This map is a homomorphism because $\mathrm{ord}_Y(f \cdot g) = \mathrm{ord}_Y(f) + \mathrm{ord}_Y(g)$.

It is also bijective because we can write down the inverse: Say we are given a divisor D, it suffices to take one of the form $D = a\,Y$ ($a \in \mathbb{Z}$ and Y is an irreducible hypersurface). We can choose a cover $\{U_i\}$ fine enough that $Y \cap U_i$ is the zero set of a $g_i \in \mathcal{O}_X(U_i)$. Set $f_i = (g_i)^a \in \mathcal{K}_X^*(U_i)$, the so defined collection is coclosed (as a $\mathcal{K}_X^*/\mathcal{O}_x^*$ - cochain): On $U_i \cap U_j$, g_i and g_j can only differ by a $\mathcal{O}_X^*(U_i \cap U_j)$ function because they define the same irreducible hypersurface. ☐

Consider now the following short exact sequence of sheaves:

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0\ . \tag{B.48}$$

As mentioned in subsections A.3.2 and B.3.1, this induces a long exact sequence in homology:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^0(X, \mathcal{K}_X^*) & \longrightarrow & H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) & \longrightarrow & H^1(X, \mathcal{O}_X^*) & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & \mathcal{K}_X(X) & \longrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \cdots \end{array} \tag{B.49}$$

We used theorem B.55 and lemma B.61.

The homomorphism $\mathcal{K}_X(X) \to \mathrm{Div}(X)$ is the assignment of a principal divisor seen in definition B.60. Apparently, there is a natural homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. A closer examination of the construction of the long exact sequence in homology shows that an $(f_i) \in H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ gets mapped to a 1-cochain $(g_{ij}) \in H^1(X, \mathcal{O}_X^*)$ with

$$g_{ij} = f_i^{-1} \cdot f_j\ , \tag{B.50}$$

this cochain is obviously coclosed.

Definition B.62. This homomorphism is called $\mathcal{O} : \mathrm{Div}(X) \to \mathrm{Pic}(X)$.

Lemma B.63. $\mathcal{O}(D + D') = \mathcal{O}(D) \otimes \mathcal{O}(D')$ *(because of how the transition functions g_{ij} look). Furthermore, $\mathcal{O}(0) = \mathcal{O}$ and $\mathcal{O}(-D) = \mathcal{O}(D)^*$.*

Theorem B.64. *From* (B.49) *we can see that a divisor D is principal if and only if $\mathcal{O}(D) = \mathcal{O}$.*

Two divisors D and D' are called linearly equivalent*, $D \sim D'$, if their difference is a principal divisor. Obviously $\mathcal{O} : \mathrm{Div}(X)/{\sim} \to \mathrm{Pic}(X)$ is an injection.*

The next question we will be concerned with is what the image of the map $\mathcal{O} : \mathrm{Div}(X)/\sim \to \mathrm{Pic}(X)$ looks like. It turns out:

Theorem B.65. *A line bundle $L \in \mathrm{Pic}(X)$ can be written as $\mathcal{O}(D)$ if and only if it admits global sections, i.e. $H^0(X, L) \neq \{0\}$.*

More specifically, for each L there is a semigroup homomorphism $Z_L : H^0(X, L)\backslash\{0\} \to \mathrm{Div}(X)$ with $\mathcal{O}(Z_L(s)) = L$. This means in particular that the difference $Z_L(s_1) - Z_L(s_2)$ is a principal divisor (theorem B.64) for $s_1, s_2 \in H^0(X, L)\backslash\{0\}$.

Proof. To prove the "⇐" part, we only have to give an explicit Z_L. The idea is that if we fix trivializations $\varphi_i : U_i \times \mathbb{C} \to L$, then the cochain consisting of $f_i = \varphi_i^{-1} \circ s|_{U_i}$ defines an element in $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ and we can use lemma B.61. For details and for "⇒" see [58, Ch. 2.3], note that the proof relies on lemma B.66 as well. □

Note. Intuitively, Z_L gives the zero set of global sections of L including multiplicity. The theorem above can be understood as follows: If the line bundle L admits global sections, we can consider the zero locus of such a section. It is a divisor D with $\mathcal{O}(D) = L$.

We want to use these results in order to rewrite the adjunction formula (theorem B.53). First we need one more lemma:

Lemma B.66. *Let D be an effective divisor. Then D is in the image of $Z_{\mathcal{O}(D)}$.*

Proof. By lemma B.61, an divisor is given by a collection of $f_i \in \mathcal{K}_X^*(U_i)$. But if D is effective, the f_i are actually holomorphic and define a global section $s \in H^0(X, \mathcal{O}(D))$ with $Z_{\mathcal{O}(D)}(s) = D$. □

Theorem B.67 (Adjunction Formula)**.** *Let Y be a smooth hypersurface, i.e. a complex submanifold of codimension 1.*

i) If Y is in the image of Z_L for some $L \in Pic(X)$, then $L|_Y \cong \mathcal{N}_{Y/X}$.

ii) In particular, $\mathcal{N}_{Y/X} \cong \mathcal{O}(Y)|_Y$ and therefore

$$K_Y \cong (K_X \otimes \mathcal{O}(Y))|_Y \ . \tag{B.51}$$

Proof. The first part of the theorem is proved by comparing the transition functions of $L|_Y$ and $\mathcal{N}_{Y/X}$.

For the second part we need to understand why $\mathcal{N}_{Y/X} \cong \mathcal{O}(Y)|_Y$, (B.51) then follows from theorem B.53. The reason is that Y seen as a divisor is effective, therefore by the previous lemma in the image of $Z_{\mathcal{O}(Y)}$. Now we can use the first part of the theorem. □

What we have seen so far is a way to assign cohomology classes $Y \in \mathrm{Div}(X)$ and $\mathcal{O}(Y) \in \mathrm{Pic}(X)$ to an irreducible hypersurface Y. Poincaré duality (see theorem A.49) gives us another way:

Definition B.68. Let Y be a smooth hypersurface. Its *fundamental class* is the Poincaré dual, denoted by $[Y] \in H^2(X)$.

Remember that its definition was $\int_Y \omega|_Y = \int_X \omega \wedge [Y]$ for all $\omega \in H^{2m-2}(X)$.

Theorem B.89 will show how these concepts are related.

B.4 Chern Classes

B.4.1 Hermitian Vector Bundles and Connections

We have already covered connections in detail in section A.5 and we have talked about the Chern connection on the tangent bundle of a complex manifold in section B.2. We can generalize this a bit, but the following results should not be surprising after the previous discussion.

Definition B.69. Let M be a differentiable manifold and $\mathbb{C}^r \to E \xrightarrow{\pi} M$ a complex vector bundle. A *hermitian structure* h on E is the smooth assignment of a hermitian scalar product h_p on every fiber $E|_p$. The pair (E, h) is called a *hermitian vector bundle*.

Example B.70. In subsection B.2.1 we have seen that a Hermitian metric g on a complex manifold induces a hermitian structure $h : (v, w) \mapsto h(v, \bar{w})$ on the holomorphic tangent bundle TX^+.

Note. One can do Hodge theory on hermitian vector bundles over a hermitian manifold (X, g) in general.

Let us quickly fix the following notation: $\Omega^{r,s}(X, E) = \Omega^{r,s}(X) \otimes E$ in analogy to e.g. $H^{r,s}_{\bar\partial}(X, E)$. Now define

$$\bar{*} : \Omega^{r,s}(X, E) \to \Omega^{m-r,m-s}(X, E^*) \tag{B.52}$$

by $\bar{*}(\omega \otimes s) = \overline{*\omega} \otimes h(s, \cdot)$. With this definition we have $\eta_1 \wedge \bar{*}\eta_2 = \langle \eta_1, \eta_2 \rangle$ vol like in subsection A.2.3, where $\langle \cdot, \cdot \rangle$ is the inner product on $\Omega^{r,s}(X) \otimes E$ induced by the hermitian metric g and the hermitian structure h. Again, this gives us a scalar product $(\eta_1, \eta_2) = \int_X \eta_1 \wedge \bar{*}\eta_2$.

All this allows us to define a Laplacian $\Delta_{\bar\partial}$ on $\Omega^{r,s}(X, E)$. We then get Hodge decomposition

$$\Omega^{r,s}_{\bar\partial}(X, E) = \bar\partial\Omega^{r,s-1}(X, E) \oplus \bar\partial^\dagger\Omega^{r,s+1}(X, E) \oplus \mathrm{Harm}^{r,s}_{\bar\partial}(X, E) \tag{B.53}$$

as well as an isomorphism

$$\mathrm{Harm}^{r,s}_{\bar\partial}(X, E) \cong H^{r,s}_{\bar\partial}(X, E) \quad \left(\cong H^s(X, E \otimes \Omega^r_X)\right) . \tag{B.54}$$

Furthermore, Poincaré duality (theorem A.49) is generalized to *Serre duality*. Serre duality is based on the map $\Lambda : H^{r,s}_{\bar\partial}(X, E) \times H^{m-r,m-s}_{\bar\partial}(X, E^*) \to \mathbb{C}$ which is bilinear and non-degenerate. Thus,

$$H^{r,s}_{\bar\partial}(X, E) \cong H^{m-r,m-s}_{\bar\partial}(X, E^*)^* . \tag{B.55}$$

This duality does not depend on g or h.

Definition B.71 (Hermitian Connection)**.** A connection D on a hermitian vector bundle (E, h) is called *hermitian* if it is compatible with the hermitian structure, i.e.

$$v[h(s_1, s_2)] = h(D_v s_1, s_2) + h(s_1, D_v s_2) . \tag{B.56}$$

Example B.72. Say we have a hermitian manifold (X, g) with induced hermitian structure h on TX^+ like in example B.70.

In general, a connection D on TX^+ induces a connection on D' on TX as follows: TX is locally $\mathbb{R}$-spanned by the basis $\{\partial_{x^\mu}, \partial_{y^\mu}\}$ and TX^+ is locally $\mathbb{R}$-spanned by the basis $\{\partial_\mu, \mathrm{i}\partial_\mu\}$. We define the isomorphism $\xi : TX \to TX^+$ as $\partial_{x^\mu} \mapsto \partial_\mu$ and $\partial_{y^\mu} \mapsto \mathrm{i}\partial_\mu$. This can also be written in a coordinate-free form:

$$\xi : TX \to TX^+, v \mapsto \frac{1}{2}\left(v - \mathrm{i}J(v)\right) . \tag{B.57}$$

This map induces the connection D' on TX by requiring $D_v(\xi \circ w) = \xi \circ D'_v(w)$.

If D is hermitian, the induced connection D' is compatible with the Riemannian metric g (in the sense of definitions A.80 and B.24).

Proof. A quick calculation shows that we can always rewrite $g(v_1, v_2)$ as

$$g(v_1, v_2) = 2\Re\left[h(\xi \circ v_1, \overline{\xi \circ v_2})\right] .$$

All that's left to do is taking the real part of both sides of the equation

$$v\, h(\xi \circ v_1, \overline{\xi \circ v_2}) = h(\xi \circ D'_v v_1, \overline{\xi \circ v_2}) + h(\xi \circ v_1, \overline{\xi \circ D'_v v_2}) . \qquad \square$$

Definition B.73. Remember that a connection D on a holomorphic vector bundle $\pi : E \to X$ is a map $D : \Omega^0(X, E) \to \Omega^1(X, E) = \Omega^{1,0}(X, E) \oplus \Omega^{0,1}(X, E)$. Hence also the connection has two components:

$$D = D^+ \oplus D^- . \tag{B.58}$$

The connection D is *compatible with the holomorphic structure* if $D^- = \bar\partial$. In local coordinates, D can then be written in the form $D = D^0 + A$ for an $A \in \Omega^{1,0}(X, \mathrm{End}(E))$.

Example B.74. We return to the setting of example B.72. By example B.50, TX^+ can be given the structure of a holomorphic vector bundle over X.

A connection D on TX^+ which is compatible with the holomorphic structure induces a connection D' on $TX^{\mathbb{C}}$ which is compatible with the complex structure in the sense of definition B.24.

Proof. By definition, the vector potential A of D is a $(1,0)$-form mapping $TX^+ \to TX^+$. Let A' be the vector potential of the induced D' (continued from TX to $TX^{\mathbb{C}}$), we know

$$A'_{\overset{(-)}{\mu}} = \xi^{-1} \circ A_{\overset{(-)}{\mu}} \circ \xi = 2\Re\, A_{\overset{(-)}{\mu}} \circ \xi = A_{\overset{(-)}{\mu}} \circ \xi + \overline{A_{\overset{(-)}{\mu}}} \circ \bar{\xi}\ .$$

Because A is a $(1,0)$-form and $\bar{A}$ a $(0,1)$-form, we see that $A'_\mu = A_\mu \circ \xi$ and $A'_{\bar{\mu}} = \bar{A}_\mu \circ \bar{\xi}$. ξ is a projector on TX^+ and A_μ maps $TX^+ \to TX^+$, also $\bar{\xi}$ is a projector to TX^- and $\bar{A}_\mu$ maps $TX^- \to TX^-$. The claim follows directly. ☐

Definition B.75 (Chern Connection)**.** On a holomorphic vector bundle E with hermitian structure h, there is exactly one connection that is compatible with both the hermitian and the holomorphic structure. It is called the *Chern connection* D.

Lemma B.76 (Curvature of the Chern Connection)**.**

i) Let D be a hermitian connection on a hermitian vector bundle. Then its curvature (seen as an $\mathrm{End}(E)$*-valued* 2*-form) is skew-hermitian, i.e.* $h(F\cdot,\cdot) = -h(\cdot, F\cdot)$ *or* $F^\dagger = -F$.

ii) Let D be a connection on a holomorphic vector bundle with $D^- = \bar{\partial}$. Then its curvature is a sum of a $(2,0)$*- and a* $(1,1)$*-form, but doesn't have a* $(0,2)$*-part.*

iii) Hence, the curvature of the Chern connection D *on a holomorphic hermitian bundle is a real, skew-hermitian* $(1,1)$*-form.*

Proof. i) We calculate locally. According to theorem A.90, $F = \mathrm{d}A + A \wedge A$. Because D is hermitian, $A^\dagger = -A$. The claim $F^\dagger = -F$ follows using $(A \wedge A)^\dagger = -A^\dagger \wedge A^\dagger$.

ii) A is a $(1,0)$-form and $F = (\bar{\partial} A) + (\partial A + A \wedge A)$. ☐

Let D again be the Chern connection on a holomorphic hermitian bundle. We already know that its curvature F_{D} is an element of $\Omega^{1,1}(X, \mathrm{End}(E))$.

According to lemma A.87, $\mathrm{d}_{\mathrm{D}} F_{\mathrm{D}} = \mathrm{d}F_{\mathrm{D}} + [A, F]_{\mathrm{D}}$ and this has to be zero because of the Bianchi identity (theorem A.91). Because A is a $(1,0)$-form and F_{D} a $(1,1)$-form we see that $\bar{\partial} F_{\mathrm{D}} = 0$. Therefore, F_{D} yields a cohomology class $[F_{\mathrm{D}}] \in H^{1,1}_{\bar{\partial}}(X, \mathrm{End}(E)) \cong H^1(X, \Omega_X \otimes \mathrm{End}(E))$.

Definition B.77 (Atiyah Class)**.** The *Atiyah class* of a holomorphic vector bundle E is a class

$$A(E) \in H^1(X, \Omega_X \otimes \mathrm{End}(E))\ . \tag{B.59}$$

Let $\varphi_i : U_i \times \mathbb{C}^r \to E$ be local trivializations and $\varphi_{ij} = \varphi_i^{-1} \circ \varphi_j$, then $A(E)$ is given by the 1-cochain $\psi_{ij} = \varphi_j \circ (\varphi_{ij}\ \mathrm{d}\varphi_{ij}^{-1}) \circ \varphi_j^{-1}$.

The Atiyah class is of interest on its own, but here we only note that

Theorem B.78. *On a holomorphic hermitian bundle, the Atiyah class* $A(E)$ *is equal to* $[F_{\mathrm{D}}]$.

B.4.2 Invariant Polynomials

Definition B.79 (Invariant Polynomial)**.** An *invariant polyomial* P is a k-multilinear symmetric map

$$P : \mathfrak{gl}(r, \mathbb{C})^k \to \mathbb{C} \tag{B.60}$$

with $P(CB_1C^{-1}, \ldots, CB_kC^{-1}) = P(B_1, \ldots, B_k)$ for all $C \in \mathrm{GL}(r, \mathbb{C})$. We'll call the space of invariant polynomials of k variables $I^k(r)$.

Given a complex vector bundle $\pi : E \to M$ of rank r, an invariant polynomial P naturally acts on $\mathrm{End}(E)$-valued forms as well:

$$P(\omega_1 \otimes s_1, \ldots, \omega_k \otimes s_k) = \omega_1 \wedge \cdots \wedge \omega_k \, P(s_1, \ldots, s_k) \ . \tag{B.61}$$

The *polarized form* $\tilde{P}$ of P is the map

$$\tilde{P} : \Omega^2(M, \mathrm{End}(E)) \to \Omega^{2k}(M),\ (\omega \otimes s) \mapsto P(\omega \otimes s, \ldots, \omega \otimes s) \ . \tag{B.62}$$

Lemma B.80.

i) A k-multilinear symmetric map P is an invariant polynomial if and only if for all $B, B_1, \ldots, B_k \in \mathfrak{gl}(r, \mathbb{C})$:

$$\sum_{j=1}^{k} P(B_1, \ldots, [B, B_j], \ldots, B_k) = 0 \ . \tag{B.63}$$

ii) Let F be the curvature of an arbitrary connection D on E. Then $\tilde{P}(F)$ is closed.

iii) If D and D' are two connections on E, then $[\tilde{P}(F_D)] = [\tilde{P}(F_{D'})]$ in $H^{2k}(M)$.

Proof. i) Follows directly from the definition using $C = \mathrm{e}^{Bt}$ and differentiating.

ii) Remembering lemma A.87, we get

$$\mathrm{d}P(F, \ldots, F) = \sum P(F, \ldots, \mathrm{d}F, \ldots, F) = \sum P(F, \ldots, \mathrm{d}_D F - [A, F], \ldots, F) = 0 \ .$$

(The first equality is just the usual Leibniz rule because d only acts on the forms.)

iii) Thanks to lemma A.78, $D' = D + A$ for a vector potential A. Let $D_t = D + tA$, its curvature F_t is $F_t = F + t\mathrm{d}_D A + t^2 A \wedge A$. We write

$$\tilde{P}(F') - \tilde{P}(F) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \tilde{P}(F_t) \ \mathrm{d}t = k \int_0^1 P(\mathrm{d}_D A + 2tA \wedge A, F_t, \ldots, F_t) \ \mathrm{d}t$$

and see by a calculation that $\mathrm{d}P(A, F_t, \ldots, F_t) = P(\mathrm{d}_D A, F_t, \ldots, F_t) + 2tP(A \wedge A, F_t, \ldots, F_t)$. This shows that

$$\tilde{P}(F') - \tilde{P}(F) = \mathrm{d} \left[k \int_0^1 P(A, F_t, \ldots, F_t) \ \mathrm{d}t \right] \ . \qquad \square$$

Definition B.81. For a complex vector bundle E of rank r, there is a map $\chi_E : I^k(r) \to H^{2k}(M)$ assigning $\chi_E(P) = [\tilde{P}(F_D)]$. As we have shown, this is independent of the connection D. The homomorphism χ_E is called *Weil homomorphism.*

Theorem B.82 (Naturality)**.** *Let f^*E be the pullback bundle defined in A.69.*

$$\chi_{f^*E} P = f^* \left(\chi_E P \right) \ . \tag{B.64}$$

Proof. If A is a vector potential on E, f^*A is a vector potential on f^*E (i.e. it satisfies (A.65)). The associated curvature can be calculated to be f^*F. This proves the claim since $\tilde{P}(f^*F) = f^*\tilde{P}(F)$. $\square$

An easy consequence is that the classes $[\chi_E(P)]$ are trivial if E is a trivial bundle, because trivial bundles are isomorphic to pullback bundles of some $\{p\} \times F$.

B.4.3 Chern Classes and Characters

Definition B.83 (Chern Classes)**.** We define invariant polynomials $P_1, \dots, P_r$ by

$$\det(\mathrm{id} + B) = 1 + \tilde{P}_1(B) + \cdots + \tilde{P}_r(B) \; . \tag{B.65}$$

($\tilde{P}_k(B)$ is the part of the expansion with degree k. Note that fixing $\tilde{P}_k(B)$ uniquely determines an invariant P_k such that $\tilde{P}_k(B) = P_k(B, \dots, B)$.)

The *k-th Chern class* $\mathrm{c}_k(E)$ is

$$\mathrm{c}_k(E) = \left[\tilde{P}_k \left(\frac{\mathrm{i}}{2\pi} F \right) \right] \in H^{2k}(M) \; . \tag{B.66}$$

The *total Chern class* is $\mathrm{c}(E) = 1 + \mathrm{c}_1(E) + \cdots + \mathrm{c}_r(E) \in H(M)$.

Definition B.84 (Chern Characters)**.** This time, we define the polynomials by

$$\mathrm{tr}(\mathrm{e}^B) = r + \tilde{P}_1(B) + \tilde{P}_2(B) + \cdots \; . \tag{B.67}$$

Then the *k-th Chern character* is $\mathrm{ch}_k(E) = \left[\tilde{P}_k \left(\frac{\mathrm{i}}{2\pi} F \right) \right] \in H^{2k}(M)$ and the *total Chern character* is $\mathrm{ch}(E) = r + \mathrm{ch}_1(E) + \mathrm{ch}_2(E) + \cdots$.

Definition B.85 (Chern Classes of a Manifold)**.** Let X be a complex manifold. We write

$$\mathrm{c}_k(X) = \mathrm{c}_k(TX^+) \tag{B.68}$$

and also $\mathrm{ch}_k(X) = \mathrm{ch}_k(TX^+)$.

Example B.86. By expanding $\det(\mathrm{id} + B) = 1 + \mathrm{tr}(B) + \cdots$ we can read off the first few Chern classes:

$$\mathrm{c}_0(E) = 1, \quad \mathrm{c}_1(E) = \frac{\mathrm{i}}{2\pi} [\mathrm{tr}\, F] \, , \quad \mathrm{c}_2(E) = -\frac{1}{2} \frac{1}{(2\pi)^2} [\mathrm{tr}\, F \wedge \mathrm{tr}\, F - \mathrm{tr}(F \wedge F)] \; . \tag{B.69}$$

For the Chern characters we find $\mathrm{ch}_k(E) = \frac{1}{k!} \mathrm{tr} \left[\frac{\mathrm{i}}{2\pi} F \right]^k$.

Lemma B.87 (Some Rules for Computing)**.**

i) On a Whitney sum bundle, the total curvature is a direct sum of the curvatures of the individual bundles. Since the determinant is multiplicative and the trace is additive, we immediately get

$$\mathrm{c}(E_1 \oplus E_2) = \mathrm{c}(E_1) \wedge \mathrm{c}(E_2) \qquad \textit{and} \qquad \mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2) \; . \tag{B.70}$$

For example, $\mathrm{c}_1(E_1 \oplus E_2) = \mathrm{c}_1(E_1) + \mathrm{c}_1(E_2)$ *and* $\mathrm{c}_2(E_1 \oplus E_2) = \mathrm{c}_2(E_1) + \mathrm{c}_2(E_2) + \mathrm{c}_1(E_1) \wedge \mathrm{c}_1(E_2)$.

ii) On a tensor product bundle $E_1 \otimes E_2$ we get $F = F_1 \otimes 1 + 1 \otimes F_2$. Therefore,

$$\mathrm{ch}(E_1 \otimes E_2) = \mathrm{ch}(E_1) \wedge \mathrm{ch}(E_2) \tag{B.71}$$

(because $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$*). A consequence is for example if E_1 and E_2 are line bundles,* $\mathrm{c}_1(E_1 \otimes E_2) = \mathrm{c}_1(E_1) + \mathrm{c}_1(E_2)$.

iii) On the dual bundle E^*, $F^* = -F^T$ and

$$\mathrm{c}_k(E^*) = (-1)^k \, \mathrm{c}_k(E) \qquad \textit{and} \qquad \mathrm{ch}_k(E^*) = (-1)^k \, \mathrm{ch}_k(E) \; . \tag{B.72}$$

iv) Finally, theorem B.82 tells us that

$$\mathrm{c}_k(f^* E) = f^* \, \mathrm{c}_k(E) \qquad \textit{and} \qquad \mathrm{ch}_k(f^* E) = f^* \, \mathrm{ch}_k(E) \; . \tag{B.73}$$

Note. An important tool is the *splitting principle*: It can be shown that there is always a ring extension $A^* \supset H^*(M)$ and elements $\gamma_i \in A^2$ such that $\mathrm{c}(E) = \prod_i (1+\gamma_i)$.

This means that the Chern classes of E essentially behave like those of $\bigoplus_i L_i$ where L_i are line bundles with chern class $\mathrm{c}(L_i) = 1 + \gamma_i$. To be more precise, we can always construct a pullback bundle $\pi^* E$ which is isomorphic to $\bigoplus_i L_i$.

In other words, *to prove a polynomial identity in the Chern classes of complex vector bundles, it suffices to prove it under the assumption that the vector bundles are the Whitney sum of complex line bundles. [63]*

This is an example for such a proof:

$$2\,\mathrm{ch}_2(E) = \frac{1}{(2\pi)^2}\,\mathrm{tr}(\mathrm{i}F)^2 = \mathrm{tr}(\mathrm{diag}(\gamma_1^2, \dots, \gamma_r^2)) = \sum_i \gamma_i^2$$

$$\mathrm{c}_1(E)^2 - 2\,\mathrm{c}_2(E) = \left(\sum_i \gamma_i\right)^2 - 2\sum_{i<j}(\gamma_i \wedge \gamma_j) = \sum_i \gamma_i^2\ .$$

Theorem B.88 (First Chern class of the Canonical Bundle)**.**

$$\mathrm{c}_1(K_X) = -\,\mathrm{c}_1(X)\ . \tag{B.74}$$

Proof. Let $\gamma_1 \dots \gamma_r$ be the Chern roots of a complex vector bundle E. Then the Chern roots of $\Lambda^q E$ are $\gamma_{i_1} + \dots + \gamma_{i_q}$ for all $1 \le i_1 < \dots < i_q \le r$. Invoking the splitting principle this shows $\mathrm{c}_1(\det E) = \mathrm{c}_1(E)$ in general. The theorem is proven by applying this to $E = \Omega_X$. □

Theorem B.89 (Chern Classes of Hypersurfaces)**.** *Let $Y \subset X$ be a smooth irreducible hypersurface and $i : Y \to X$ the inclusion.*

i) In that case,

$$i^*\,\mathrm{c}(X) = \mathrm{c}(Y) \wedge i^*\,\mathrm{c}(\mathcal{O}(Y))\ . \tag{B.75}$$

By expanding, we get in particular $\mathrm{c}_1(Y) = i^*\left(\mathrm{c}_1(X) - \mathrm{c}_1(\mathcal{O}(Y))\right)$.

ii) Let $[Y] \in H^2(X)$ be the fundamental class of Y (see definition B.68). Then

$$\mathrm{c}_1(\mathcal{O}(Y)) = [Y]\ . \tag{B.76}$$

Because of linearity, this also holds for divisors $D = \sum_i a_i Y_i$ in general: $\mathrm{c}_1(\mathcal{O}(D)) = \sum_i a_i [Y_i]$.

Proof. i) We rewrite the normal bundle sequence (B.42) using the result $\mathcal{N}_{Y/X} \cong \mathcal{O}(Y)|_Y$ from theorem B.67. Now it looks like

$$0 \to TY^+ \to i^*TX^+ \to i^*\mathcal{O}(Y) \to 0\ . \tag{B.77}$$

The claim follows directly from the fact that for a short exact sequence $0 \to E \to F \to G \to 0$ of holomorphic vector bundles,

$$c(F) = c(E) \wedge c(G)\ . \tag{B.78}$$

The reason is that we can view the sequence as a short exact sequence of complex vector bundles. Such a sequence always splits, i.e. $F \cong E \oplus G$ with a smooth isomorphism. Chern classes only see the complex and not the holomorphic structure, this finishes the proof.

Note that a short exact sequence of complex vector bundles always splits because we can use a partition of unity. In general there is no holomorphic partition of unity though, so there might be no holomorphic splitting.

ii) The proof is highly non-trivial. The basic idea is to choose some hermitian structure on the line bundle $\mathcal{O}(Y)$, let F be the curvature of its Chern connection. Now we have to prove

$$\frac{\mathrm{i}}{2\pi}\int_X F \wedge \omega = \int_Y \omega$$

for all closed real forms α. The left hand side is first converted into an integral over the boundary of D_ε using Stokes' theorem A.45, where D_ε is a kind of ε-neighborhood of Y. After some more work we can then use the residue theorem to prove the claim. See [58, Prop. 4.4.13]. □

Note. On a line bundle L, we have several definitions of the first Chern class:

- The one we have used above, $\mathrm{c}_1(L) = \frac{\mathrm{i}}{2\pi}[\tilde{P}_1(F)]$.
- As we have seen, this is equal to

$$\mathrm{c}_1(L) = \frac{\mathrm{i}}{2\pi}[F_\mathrm{D}] = \frac{\mathrm{i}}{2\pi}A(L) \ , \tag{B.79}$$

  where D is the Chern connection and $A(L)$ the Atiyah class (definition B.77) of the line bundle.
- Another possibility is as follows: The *exponential sequence*

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \xrightarrow{f\mapsto \exp(2\pi\mathrm{i}\, f)} \mathcal{O}_X^* \to 0 \tag{B.80}$$

  is a short exact sequence of sheaves. This induces a long exact sequence in homology,

$$\cdots \to \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \underline{\mathbb{Z}}) \to \cdots \ .$$

  There is a natural homomorphism from $H^2(X, \underline{\mathbb{Z}})$ to $H^2(X, \underline{\mathbb{C}}) = H^2(X)$ and it turns out that

$$c_1(L) = -\delta(L) \ . \tag{B.81}$$

Definition B.90. The *canonical class* $\mathcal{K}_X$ of X is $\mathcal{K}_X = \mathrm{c}_1(K_X)$, and the anticanonical class is $\bar{\mathcal{K}}_X = \mathrm{c}_1(\bar{K}_X)$. Theorems B.88 and B.89 tell us that

$$\bar{\mathcal{K}}_X = \mathrm{c}_1(X) \tag{B.82}$$

B.4.4 Calabi-Yau Manifolds

Note. For literature, see also [59, Ch. 1.2], [22, Ch. 14] and [64, Ch. 9].

Motivation. From a physics perspective, we are interested in compact complex manifolds. They are used for *string compactification* where the e.g. 10-dimensional spacetime $\mathcal{M}_{10}$ on which the string degrees of freedom live is of product form: $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \mathcal{M}_6$. The "internal" compact manifold $\mathcal{M}_6$ determines the effective theory in $\mathbb{R}^{1,3}$ at low energies.

We want this effective theory to have $\mathcal{N} = 1$ *supersymmetry*. It can be shown that a necessary condition for supersymmetry is the existence of a globally defined, covariant constant spinor field on $\mathcal{M}_6$ (see section 3.3). A necessary condition for this in turn is that $\mathcal{M}_6$ is a Ricci-flat Kähler manifold.

On the other hand, there is a sufficient condition for $\mathcal{N} = 1$ supersymmetry involving the *holonomy* of $\mathcal{M}_6$. We have not defined holonomy yet but will make up for this in definition B.91. The mentioned sufficient condition is that the holonomy group of $\mathcal{M}_6$ (which is a 3-complex dimensional manifold, as we know now) is exactly SU(3). We will now investigate the relation between those conditions and in passing define Calabi-Yau manifolds.

Definition B.91 (Holonomy)**.** Let $\pi : E \to M$ be a vector bundle, D a connection on it and $p \in M$. A loop $\gamma : [0,1] \to M$ based at p (i.e. $\gamma(0) = \gamma(1) = p$) defines an element $G_\gamma \in \mathrm{GL}(E|_x) : G_\gamma$ maps a vector $v \in E|_x$ to the result of parallel transporting v around the loop γ with respect to D.

The *(global) holonomy group* of D is

$$\mathrm{Hol}(D) = \{G_\gamma\} \subset \mathrm{GL}(\dim E) \ , \tag{B.83}$$

this is (up to an isomorphism) independent of $p \in M$.

The *local holonomy group* $\mathrm{Hol}_p^*(D)$ is the subgroup coming from infinitesimal contractible loops γ (i.e. the direct limit of making those loops smaller and smaller). The local holonomy group is contained in the connected component of $\mathrm{Hol}(D)$ containing the identity.

Note. We define the holonomy of a Riemannian manifold (M, g) to be that of its Levi-Civita connection. M is orientable if and only if its holonomy group is contained in $\mathrm{SO}(\dim M)$.
M admits a complex structure such that (M, g) is Kähler if and only if it is even dimensional and its holonomy group is contained in $\mathrm{U}(\dim M/2)$.

Lemma B.92. *Let X be a compact Kähler manifold of complex dimension m. Then the following conditions are equivalent:*

i) X admits a Kähler metric such that the local holonomy is contained in $\mathrm{SU}(m)$.

ii) X admits a Ricci-flat Kähler metric.

iii) The first Chern class $\mathrm{c}_1(X)$ vanishes.

Some authors then call X a Calabi-Yau manifold.

Proof. For the equivalence between i) and ii), we consider a vector $v \in T_pX^+$ and an infinitesimal loop γ based at p. Under parallel transport it changes as

$$\Delta v^\rho = \left(\oint_\gamma x^\mu \, \mathrm{d}\bar{x}^\nu \right) R^\rho_{\sigma\, \mu\bar{\nu}} v^\sigma \ .$$

Having used theorem B.35 we see that holomorphic vectors are only mapped to holomorphic vectors[2] and the local holonomy must be contained in $\mathrm{U}(m)$ (as noted above). The trace of the matrix acting on v corresponds to the $\mathfrak{u}(1)$ part in $\mathfrak{u}(m) = \mathfrak{su}(m) \oplus \mathfrak{u}(1)$. According to theorem B.35 it vanishes if and only if $\mathrm{Ric} = 0$.

The implication ii) $\Rightarrow$ iii) is also clear from theorem B.35. The other direction is very non-trivial, it is the content of the Calabi-Yau theorem [58, Cor. 4.B.22]. ☐

We know that $\mathrm{c}_1(K_X) = -\mathrm{c}_1(X)$ (theorem B.88), so if $\mathrm{c}_1(X) = 0$ then also $\mathrm{c}_1(K_X) = 0$. This does not imply that K_X is trivial, but if on the other hand K_X is trivial then $\mathrm{c}_1(X) = 0$.

Lemma B.93. *Let X be a compact Kähler manifold of complex dimension m. The following conditions are equivalent and stronger*[3] *than the conditions in lemma B.92:*

i) K_X is trivial.

ii) X has a holomorphic m-form that vanishes nowhere.

iii) X admits a Kähler metric such that the global holonomy is contained in $\mathrm{SU}(m)$.

Some authors then call X a Calabi-Yau manifold.

Proof. The equivalence i) $\Leftrightarrow$ ii) is a direct consequence of example B.49.

The implication iii) $\Rightarrow$ ii) can be seen by explicitly constructing such a holomorphic m-form: Just take a m-vector ω in $\Lambda^m TX^+$ at some point p. The form at point q is ω parallel transported from p to q, this is independent of the path because ω transforms trivially under $\mathrm{SU}(m)$.

On the other hand, a holomorphic m-form must always be covariantly constant. Such a form can only by defined globally if the global holonomy is contained in $\mathrm{SU}(m)$, because otherwise it doesn't transform as a singlet. ☐

We use an even more restrictive definition of a Calabi-Yau manifold, because we want a compactification on a Calabi-Yau manifold to yield $\mathcal{N} = 1$ supersymmetry:

Definition B.94 (Calabi-Yau Manifold)**.** A *Calabi-Yau manifold* or *CY m-fold* is a compact Kähler manifold with global holonomy equal to $\mathrm{SU}(m)$. [65]

Theorem B.95. *In a manifold like in B.93,*

[2] This was also already clear because the Chern connection is compatible with the complex structure.
[3] If X is simply connected, they are equivalent.

i) $b^{m,0} = b^{0,m} = 1$.

ii) $b^{r,0} = b^{m-r,0}$ *and* $b^{0,s} = b^{0,m-s}$.

iii) If the manifold is CY (according to the definition above), additionally $b^{r,0} = 0$ *for* $0 < r < m$.

Proof. i) The nowhere vanishing holomorphic m-form is unique: Two such forms Ω and Ω' would be related by $\Omega' = f\Omega$ with f holomorphic. But, as noted after definition B.6, a holomorphic function on a compact manifold is constant.

ii) Let $\alpha = \frac{1}{r!}\alpha_{\mu_1\dots\mu_r}\,\mathrm{d}z^{\mu_1}\wedge\dots\wedge\mathrm{d}z^{\mu_r}$ be a $\bar\partial$-harmonic $(r,0)$-form (see theorem B.37). Define now the $(0,m-r)$-form β by $\beta_{\bar\nu_{r+1}\dots j_m} = \frac{1}{(m-r)!}\bar\Omega_{\bar j_1\dots\bar j_m}\alpha^{\bar j_1\dots\bar j_r}$. One can show that β is $\bar\partial$-harmonic and this is an isomorphism.

iii) Take a harmonic representative α of $H^{r,0}_{\bar\partial}(X)$. From the facts that X is Ricci-flat and compact, one shows that α needs to be covariantly constant. Hence, it should transform as a singlet under the holonomy group, but it transforms as the $\Lambda^r\underline{m}$ which only contains a singlet if $r = 0$ or $r = m$. ☐

Example B.96 (CY Threefold)**.** Combining this with theorem B.40, the Hodge diamond of e.g. a Calabi-Yau threefold looks as follows:

$$\begin{pmatrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & b^{1,1} & & 0 & \\ 1 & & b^{2,1} & & b^{2,1} & & 1 \\ & 0 & & b^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{pmatrix}. \tag{B.84}$$

Example B.97 (CY 4-Fold)**.** We will also need the Hodge diamond of a CY 4-fold, it is

$$\begin{pmatrix} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & b^{1,1} & & 0 & & \\ & 0 & & b^{2,1} & & b^{2,1} & & 0 & \\ 1 & & b^{3,1} & & b^{2,2} & & b^{3,1} & & 1 \\ & 0 & & b^{2,1} & & b^{2,1} & & 0 & \\ & & 0 & & b^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{pmatrix}. \tag{B.85}$$

There are only three independent Hodge numbers because (without proof)

$$b^{2,2} = 2\left(22 + 2b^{1,1} + 2b^{3,1} - b^{2,1}\right)\ . \tag{B.86}$$

Appendix C

Projective Geometry

Note. Literature: [58], [59, Ch. 1 - 6] and [48, Ch. 1.3].

C.1 Basics

We have already defined defined complex projective spaces in example B.4. In example B.31 we saw that they are Kähler manifolds. Furthermore, we already know that all projective manifolds are Kähler manifolds (see theorem B.33).

In this chapter we will study projective spaces in much more detail. We will use the notation

$$\mathbb{P}^n = \mathbb{C}P^n \ . \tag{C.1}$$

Let's start by stating a few results about their cohomology:

Theorem C.1 (Hodge Diamond)**.** *The Hodge numbers of $\mathbb{P}^n$ are $b^{p,q} = 1$ if $p = q \leq n$ and $b^{p,q} = 0$ otherwise. For example, the Hodge diamond of $\mathbb{P}^2$ looks like this:*

$$\begin{pmatrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{pmatrix} . \tag{C.2}$$

Proof. With the methods shown in subsection A.3.2, one can calculate the homology of $\mathbb{P}^n$: One triangulates $\mathbb{P}^1$ and then gets the homology of $\mathbb{P}^n$ by induction, using the Mayer-Vietoris sequence [57]. The result is that $H_{2k}(\mathbb{P}^n, R) = R$ (for $0 \leq k \leq n$ and $R \in \{\mathbb{Z}, \mathbb{R}\}$) and all others are zero.

We see that $b_{2k} = 1$ for $0 \leq k \leq n$ and $b_{2k+1} = 0$. That means that the rows in the Hodge diamond have to add up to 1 or 0 alternatingly. Because of the symmetries (theorem B.37), this is only possible if the Hodge diamond looks like claimed above. □

Lemma C.2. *The cohomology classes $H^{2k}(\mathbb{P}^n, \mathbb{Z}) = H^{k,k}_{\bar\partial}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ are generated by the fundamental classes of $(n-k)$-dimensional planes $\mathbb{P}^{n-k} \subset \mathbb{P}^n$.*

This means that if $[V] \in H^{2k}(\mathbb{P}^n, \mathbb{Z})$, then $[V] = d\,[\mathbb{P}^{n-k}]$ for some $d \in \mathbb{Z}$. We usually just write $[V] = d$.

C.1.1 The Tautological Line Bundle

Definition C.3. We first define the holomorphic line bundle

$$\mathcal{O}(-1) = \{(\ell, z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : z \in \ell\} \tag{C.3}$$

(the notation $\ell \in \mathbb{P}^n$ hints at the fact that a equivalence class in $\mathbb{P}^n$ is a line in $\mathbb{C}^{n+1}$).

The projection is simply $\pi : (\ell, z) \mapsto \ell$. Over a chart[1] U_i we have a canonical trivialization $\varphi_i^{-1} : \mathcal{O}(-1)|_{U_i} \to U_i \times \mathbb{C}$ given by $\varphi_i^{-1}(\ell, z) = (\ell, z_i)$. This means that on $U_i \cap U_j$, the transition functions are $g_{ij} = \frac{z_i}{z_j}$.

Definition C.4 (Tautological Line Bundle)**.** The dual bundle of $\mathcal{O}(-1)$ is the *tautological line bundle* $\mathcal{O}(1)$. We define $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ for $k > 0$, $\mathcal{O}(k) = \mathcal{O}(-1)^{\otimes(-k)}$ for $k < 0$ and $\mathcal{O}(0) = \mathcal{O}$. This is a homomorphism $\mathcal{O} : \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n)$.

Note. By definition, $\mathcal{O}(-1)$ is embedded in $\mathcal{O}^{\oplus(n+1)} = \mathbb{P}^n \times \mathbb{C}^{n+1}$. A coordinate z_i can be seen as a linear map from $\mathcal{O}^{\oplus(n+1)}$ to $\mathcal{O}$. Restricting it to $\mathcal{O}(-1)$ it is also a holomorphic map from $\mathcal{O}(-1) \to \mathcal{O}$. Consequently it is a global section of $\mathcal{O}(-1)^* = \mathcal{O}(1)$.

In more generality, $\mathcal{O}(-k)$ can be embedded in $\mathbb{P}^n \times \left(\mathbb{C}^{n+1}\right)^{\otimes k}$. A polynomial $P_s \in \mathbb{C}[z_0 \dots z_n]_k$ of the coordinates that is homogeneous of degree k gives rise to a holomorphic map from $\mathcal{O}(-k) \to \mathcal{O}$. We associate to it a global holomorphic section s of $\mathcal{O}(k)$.

Theorem C.5 (Sections of $\mathcal{O}(k)$)**.** *For $k \geq 0$, the space $\mathbb{C}[z_0 \dots z_n]_k$ is canonically isomorphic to the global sections of $\mathcal{O}(k)$, $H^0(\mathbb{P}^n, \mathcal{O}(k))$.*

Proof. The given map $\mathbb{C}[z_0 \dots z_n]_k \to H^0(\mathbb{P}^n, \mathcal{O}(k))$ is obviously linear. It is also injective because $\pi_2 : \mathcal{O}(-k) \to \left(\mathbb{C}^{n+1}\right)^{\otimes k}$ is surjective – therefore, a polynomial inducing the trivial map has to be zero.

The idea for proving surjectivity is to first fix a global holomorphic section s of $\mathcal{O}(k)$ induced by a polynomial P_s. Given another global holomorphic section t, the quotient t/s is a meromorphic function on $\mathbb{P}^n$. Let $\pi : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ and $G = P_s \cdot (\pi \circ (t/s))$. G can be continued to a holomorphic function on $\mathbb{C}^{n+1}$. It is also obviously of degree k, hence a polynomial. G is mapped to t which proves surjectivity. Details: [58, Prop. 2.4.1] or [48, Ch. 1.3]. □

C.1.2 Divisors

Definition C.6 (Algebraic Variety)**.** An *algebraic variety* $V \subset \mathbb{P}^n$ is the zero set (*locus*) of a collection of homogeneous polynomials $\{F_\alpha(z_0, \dots, z_n)\}$.

Note. It makes sense to talk about the vanishing locus of a homogeneous polynomial in the coordinates because $F(z_0, \dots, z_n) = 0$ if and only if $F(\lambda z_0, \dots, \lambda z_n) = 0$. Equivalently, the zero set of the polynomial can be described as the zero set of the corresponding global section of $\mathcal{O}(\deg f)$.

Note. The zero locus of a polynomial F in projective space is usually written as $V(F)$.

Theorem C.7 (Chow)**.** *Obviously, algebraic varieties are analytic subvarieties of $\mathbb{P}^n$.*

On the other hand, any analytic subvariety of $\mathbb{P}^n$ is algebraic.

Proof. Omitted, see [48, Ch. 1.3]. □

Definition C.8 (Degree of a Hypersurface)**.** Let Y be an irreducible hypersurface, by theorem C.7 it is given as the zero locus of an irreducible homogeneous polynomial of degree d. The degree $\deg Y$ of the hypersurface is equal to the degree of the polynomial:

$$\deg Y = d \, . \tag{C.4}$$

Note. Irreducibility of the polynomial is crucial here, otherwise the value of the degree could not be unique. If we allow reducible polynomials, we have to consider its zero locus as a divisor instead of just as an algebraic variety: Let F be irreducible with zero locus Y, then the zero locus of F^2 is $2Y$. This is what the map Z_L defined in theorem B.65 does.

Theorem C.9 (Divisors of Projective Space)**.**

i) Up to linear equivalence, $\mathrm{Div}(\mathbb{P}^n)$ *has only one generator:*

$$\mathrm{Div}(\mathbb{P}^n)/\!\sim\, \cong \mathbb{Z} \, . \tag{C.5}$$

[1] We will name the charts U_i and the homogeneous coordinates z_i, differing from U_μ and z^μ in example B.4.

ii) The isomorphism is given by $\deg : \mathrm{Div}(\mathbb{P}^n) \to \mathbb{Z}, \sum_i a_i Y_i \mapsto \sum_i a_i \deg(Y_i)$ *which factorizes over* $\sim$.

iii) Let Y *be a hyperplane, meaning that* Y *is an irreducible hypersurface of degree* 1 *(in other words* $Y \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$*). Then,* $\mathcal{O}(Y) = \mathcal{O}(1)$ *such that*

$$\begin{tikzcd} \mathrm{Div}(\mathbb{P}^n)/\sim \arrow[rr, "\deg", "\sim"'] \arrow[dr, "\mathcal{O}"'] & & \mathbb{Z} \arrow[dl, "\mathcal{O}"] \\ & \mathrm{Pic}(\mathbb{P}^n) & \end{tikzcd} \quad \textit{commutes.} \tag{C.6}$$

Proof. Parts i) and ii) follow from the insight that a divisor D is principal if and only if $\deg D = 0$. The proof of this fact is somewhat technical, we will skip it here.

For part iii) consider e.g. the hyperplane $Y = (z_0)$. On a chart U_i it is the zero set of the function $f_i = \frac{x_0}{x_i} \in \mathcal{K}^*_{\mathbb{P}^n}(U_i)$. This cochain is by lemma B.61 the corresponding element in $H^0(\mathbb{P}^n, \mathcal{K}^*_{\mathbb{P}^n}/\mathcal{O}^*_{\mathbb{P}^n})$. By definition, $\mathcal{O}(Y)$ is the line bundle with transition functions $f_i^{-1} \cdot f_j = \frac{z_j}{z_i}$, this is $\mathcal{O}(1)$ by definition C.4. □

Note. This gives us an understanding for theorem C.7 in the special case where the analytic subvariety is a hypersurface Y: Y is an effective divisor, therefore Y is in the image of $Z_{\mathcal{O}(Y)}$ by lemma B.66. By theorem C.9, $\mathcal{O}(Y)$ is of the form $\mathcal{O}(k)$ the sections of which are homogeneous polynomials according to theorem C.5.

C.1.3 Chern Classes and Line Bundles

Theorem C.10 (Canonical Bundle of Projective Space)**.**

$$K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)\ . \tag{C.7}$$

Proof. Calculation shows that the transition functions of $K_{\mathbb{P}^n}$ are $g_{ij} = (-1)^{i-j} \left(\frac{z_i}{z_j}\right)^{n+1}$. This is in the same cochain class like the transition functions $\tilde{g}_{ij} = \left(\frac{z_i}{z_j}\right)^{n+1}$ of $\mathcal{O}(-n-1)$. □

As mentioned before, for a short exact sequence $0 \to E \to F \to G \to 0$ of holomorphic vector bundles, $\det F \cong \det E \otimes \det G$. Therefore we could have proven theorem C.10 as well with the *Euler sequence*:

Theorem C.11 (Euler Sequence)**.** *On* $\mathbb{P}^n$ *there exists the following short exact sequence of holomorphic vector bundles:*

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus(n+1)} \to T(\mathbb{P}^n)^+ \to 0 \tag{C.8}$$

where the inclusion $\mathcal{O} \to \mathcal{O}(1)^{\oplus(n+1)}$ *is the twisted version of the inclusion* $\mathcal{O}(-1) \subset \mathcal{O}^{\oplus(n+1)}$ *we have already seen above.*

Proof. Omitted, see [58, Prop. 2.4.4] □

We now come to the Chern classes of the line bundles $\mathcal{O}(k)$. Let $Y \subset \mathbb{P}^n$ be a hyperplane, then we know from theorem C.9 that $\mathcal{O}(Y) = \mathcal{O}(1)$. Theorem B.89 tells us directly that

$$\mathrm{c}_1(\mathcal{O}(1)) = [Y] = 1\ . \tag{C.9}$$

Definition C.12. In the following, we will call this fundamental class J:

$$J = \mathrm{c}_1(\mathcal{O}(1))\ . \tag{C.10}$$

As before, we will set $J = 1$ when it is clear from the context.

Theorem C.13. *The first Chern classes of the Bundles $\mathcal{O}(k)$ are*

$$\mathrm{c}_1(\mathcal{O}(k)) = \mathrm{c}_1\left(\mathcal{O}(1)^{\otimes k}\right) = k\ \mathrm{c}_1(\mathcal{O}(1)) = k\ . \tag{C.11}$$

Because $\mathcal{O}(k)$ are line bundles, this means $c(\mathcal{O}(k)) = 1 + kJ$.

In particular, theorem C.10 implies

$$c_1(\mathbb{P}^n) = -c_1(K_{\mathbb{P}^n}) = -c_1(\mathcal{O}(-n-1)) = n+1\ . \tag{C.12}$$

Note. We can explicitly compute $c_1(\mathcal{O}(1))$. Remember that $c_1(L) = \frac{2\pi}{\mathrm{i}}[F]$ for F the curvature of any connection on L.

A hermitian structure on a line bundle is just a scalar positive function h. In this case, the curvature of the Chern connection can be calculated to be $F_{\mathrm{D}} = -\partial\bar{\partial}\log(h)$ [58, Ex. 4.3.9].

On line bundles, there exists a natural hermitian structure [58, Ex. 4.1.2]. For $\mathcal{O}(1)$ it is given on the chart U_i by $h[z_0 : \cdots : z_n] = \sum_j \left|\frac{z_j}{z_i}\right|^2$. Comparing this to the Kähler form ω_{FS} of example B.31 this shows that

$$F_{\mathrm{D}} = \frac{2\pi}{\mathrm{i}}\omega_{FS}\ . \tag{C.13}$$

Explicit calculation shows easily that $\int_{\mathbb{P}^1}\omega_{FS} = 1$, and therefore

$$\int_{\mathbb{P}^1} J = 1\ . \tag{C.14}$$

It is possible to define Chern classes axiomatically, (C.14) then serves as the *normalization axiom*.[2]

Now we have found the total Chern classes of $\mathcal{O}(k)$ and the first Chern class of $\mathbb{P}^n$. We are still missing the total Chern class of $\mathbb{P}^n$, however.

Theorem C.14 (Total Chern Class of Projective Space)**.**

$$c(\mathbb{P}^n) = c(\mathcal{O}(1))^{n+1} = (1+J)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}J^k\ . \tag{C.15}$$

Proof. The proof is based on the Euler sequence (theorem C.11). We already used in the proof of B.89 that for $0 \to E \to F \to G \to 0$, $c(F) = c(E) \wedge c(G)$. Applying this to the Euler sequence proves the claim. □

We close this subsection by finishing our analysis of line bundles: The line bundles $\mathcal{O}(k)$ are, in fact, all holomorphic line bundles over projective space.

Theorem C.15 (Line Bundles of Projective Space)**.**

$$\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z} \tag{C.16}$$

and $\mathcal{O} : \mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n)$ is an isomorphism.

Proof. We first prove that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. We use the exponential sequence $0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}^*_{\mathbb{P}^n} \to 0$ (see (B.80)). It induces a long sequence in homology:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) & \longrightarrow & H^1(\mathbb{P}_n, \mathcal{O}^*_{\mathbb{P}^n}) & \xrightarrow{c_1} & H^2(\mathbb{P}^n, \mathbb{Z}) & \longrightarrow & H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) & \longrightarrow \cdots \\
& \| & & \| & & \| & & \| & \\
& H^{0,1}_{\bar{\partial}}(\mathbb{P}^n) = 0 & \longrightarrow & \mathrm{Pic}(\mathbb{P}^n) & \longrightarrow & \mathbb{Z} & \longrightarrow & H^{0,2}_{\bar{\partial}}(\mathbb{P}^n) = 0 &
\end{array} \tag{C.17}$$

and its exactness proves the claim.

From $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ we immediately get that $\mathcal{O}$ is injective. In order to prove surjectivity we claim that, for $L \in \mathrm{Pic}(\mathbb{P}^n)$, if $\mathrm{c}_1(L) = d$ then $\mathcal{O}(d) = L$. This follows from theorem B.89 and from injectivity of c_1, because $\mathrm{c}_1(\mathcal{O}(d)) = d = \mathrm{c}_1(L)$. □

[2] The other axioms are $c_0(E) = 1$, naturality (lemma B.87 v)) and the sum formula (lemma B.87 i)).

C.1.4 Complete Intersections

Definition C.16 (Degree of an Algebraic Variety)**.** Let $V \subset \mathbb{P}^n$ be an algebraic variety given by k polynomials. Its fundamental class $[V]$ lies in $H^{2k}(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$ (see theorem C.1).

According to lemma C.2, $[V] = d$ for some $d \in \mathbb{Z}$. We call d the *degree* of the variety:

$$\deg V = d \, . \tag{C.18}$$

Lemma C.17. *Definitions C.8 and C.16 agree.*

Proof. Let Y be a hypersurface, we know from theorem C.9 that $\mathcal{O}(Y) = \mathcal{O}(\deg Y)$ (using definition C.8). Therefore, using theorems B.89 and C.13, $[Y] = \mathrm{c}_1(\mathcal{O}(Y)) = \mathrm{c}_1(\mathcal{O}(\deg Y)) = \deg Y$. □

Theorem C.18 (Canonical Bundle of a Hypersurface)**.** *Let $Y \subset \mathbb{P}^n$ be a hypersurface of degree d. Then*

$$K_Y = \mathcal{O}(d - n - 1)|_Y \, . \tag{C.19}$$

Proof. The claim follows directly from the adjunction formula B.67, because we know $\mathcal{O}(Y) = \mathcal{O}(d)$ and $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$. □

Theorem C.19 (Chern Class of a Hypersurface)**.** *We directly see $\mathrm{c}_1(Y) = n + 1 - d$. The total Chern class is*

$$\mathrm{c}(Y) = \frac{(1+J)^{n+1}}{1+dJ} \, . \tag{C.20}$$

The fraction is to be understood in a symbolic way using Taylor expansion.

Proof. This follows directly from theorem B.89. □

What we have done so far for hypersurfaces can be extended to so-called complete intersections:

Definition C.20 (Complete Intersection)**.** Let $F_1, \dots, F_k$ be irreducible homogeneous polynomials of degrees $\deg(F_i)$ in the coordinates $z_0, \dots, z_n$. Let X be the locus in $\mathbb{P}^n$ given by the intersection of their zero sets,

$$X = V(F_1) \cap \dots \cap V(F_k) \, . \tag{C.21}$$

We further require that 0 is a regular value of $(f_1, \dots, f_k) : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}^k$. Then X is a complex submanifold of dimension $n - k$, we say that X is a *complete intersection.*

Alternatively, a complete intersection is an algebraic variety V of dimension k where the ideal of homogeneous polynomials vanishing on V is generated by exactly $(n - k)$ elements.

Theorem C.21 (Canonical Bundle of a Complete Intersection)**.** *Let $V = V(F_1) \cap \dots \cap V(F_k) \subset \mathbb{P}^n$ be a complete intersection and $q_a = \deg F_a$ the degrees of the defining polynomials. Then its canonical bundle is*

$$K_V = \mathcal{O}\left(\sum_{a=1}^{k} q_a - n - 1\right)\Bigg|_V \, . \tag{C.22}$$

Theorem C.22 (Chern Class of a Complete Intersection)**.** *We directly see $\mathrm{c}_1(V) = n + 1 - \sum_{\alpha=1}^{k} q_a$. The total Chern class is*

$$\mathrm{c}(V) = \frac{(1+J)^{n+1}}{\prod_{\alpha=1}^{k}(1 + q_a J)} \, . \tag{C.23}$$

Proof. One way to understand this is that V is the zero locus of a section of $\bigoplus_{a=1}^{k} \mathcal{O}(q_a)$ and the Chern class of a direct sum is the (wedge) product of the individual Chern classes. □

C.1.5 Projective Calabi-Yau 3-Folds

From the physics point of view, we are mostly interested in Calabi-Yau 3-folds. Here we will try to construct some which are submanifolds of projective space. The first attempt could be to realize the CY as a hypersurface in projective space – in order to get a 3-fold we take $\mathbb{P}^4$. According to theorem C.19, the Calabi-Yau condition is satisfied for $d = 5$:

Theorem C.23. *A quintic hypersurface in $\mathbb{P}^4$ is Calabi-Yau.*

We follow the notation of [59] and denote it by $\left[4 \,\middle\|\, 5\right]$.

Definition C.24 (Configuration Matrix)**.** The family of complete intersections in $\mathbb{P}^n$ defined by polynomials of degrees $q_1, \ldots, q_k$ is denoted by the *configuration matrix*

$$\left[n \,\middle\|\, q_1 \,\cdots\, q_k\right] \ . \tag{C.24}$$

Note. The reason for this notation will become clear in section C.2.

Theorem C.25. *A Calabi-Yau 3-fold can be realized as a complete intersection in projective space in five ways:*

1. *A quintic in $\mathbb{P}^4$: $\left[4 \,\middle\|\, 5\right]$.*
2. *An intersection of a quadric and a quartic in $\mathbb{P}^5$: $\left[5 \,\middle\|\, 2\ 4\right]$.*
3. *An intersection of two cubics in $\mathbb{P}^5$: $\left[5 \,\middle\|\, 3\ 3\right]$.*
4. *An intersection of two quadrics and a cubic in $\mathbb{P}^6$: $\left[6 \,\middle\|\, 2\ 2\ 3\right]$.*
5. *An intersection of four quadrics in $\mathbb{P}^7$: $\left[7 \,\middle\|\, 2\ 2\ 2\ 2\right]$.*

Proof. It is easy to see in the configuration matrix: We want a threefold, so the number of columns in the right part needs to be $n - 3$. For the CY condition, those columns have to sum up to $n + 1$. Finally, the entries need to be natural numbers greater than 1 (because e.g. $\left[5 \,\middle\|\, 1\ 5\right] = \left[4 \,\middle\|\, 5\right]$). ☐

Definition C.26 (Euler Characteristic)**.** The *Euler characteristic* χ_E of a manifold M is given by

$$\chi_E(M) = \sum_{q=0}^{\dim M} (-1)^q b^q \ . \tag{C.25}$$

For a CY 3-fold X this means $\chi_E(X) = 2(b^{1,1} - b^{2,1})$ (see the Hodge diamond in example B.96).
For a CY 4-fold Y this means $\chi_E(Y) = 6(b^{1,1} + b^{3,1} - b^{2,1})$ (see the Hodge diamond in example B.97).

Note. For a simplicial complex K, let I_q be the number of q-simplexes in the complex. Then we define $\chi_E(K) = \sum_{q=0}^{\dim K} (-1)^q I_q$ which is a generalization of the formula "vertices − edges + faces" for polyhedra. The *Euler-Poincaré theorem* shows that those definitions agree [54, Thm. 3.7].

Lemma C.27 (Calculating the Euler Characteristic)**.** *Let $V \subset \mathbb{P}^n$ be a CY 3-fold.*

i) The Gauss-Bonnet theorem tells us that $\chi_E(V) = \int_V \mathrm{c}_3(V)$ [54, Ch. 11.4.2]

ii) It is important to remember that $\int_V \omega = \int_{\mathbb{P}^n} [V] \wedge \omega$.

iii) (C.14) *can be generalized to $\int_{\mathbb{P}^n} J^n = 1$.*

Example C.28 (Euler Characteristic of the Quintic)**.** Let $V \in \left[4 \,\middle\|\, 5\right]$. We need its third Chern class and its fundamental class.

$\mathrm{c}_3(V)$ can be calculated from theorem C.19, expanding the fraction and taking the J^3 terms. The result is $\mathrm{c}_3(V) = \left[\binom{5}{3} - 5\binom{5}{2} + 25\binom{5}{1} - 125\binom{5}{0}\right] J^3 = -40J^3$.

For the fundamental class we remember that $[V] = \mathrm{c}_1(\mathcal{O}(V)) = \mathrm{c}_1(\mathcal{O}(5)) = 5J$.

Thus

$$\chi_E(\left[4 \,\middle\|\, 5\right]) = \int_{\mathbb{P}^4} 5J \wedge (-40J^3) = -200 \ . \tag{C.26}$$

Example C.29. Let's also calculate the Euler characteristic of $V \in \left[5 \,\middle\|\, 2\ 4\right]$. The third Chern class is $\mathrm{c}_3(V) = -22J^3$.

Theorem B.89 is at first glance insufficient for calculating $[V]$ because it is only valid for hypersurfaces. However, we can apply it iteratively: V is the zero set of a section of $\mathcal{O}(2) \oplus \mathcal{O}(4)$ and hence its fundamental class is $[V] = 2J \wedge 4J = 8J^2$.

Resulting in

$$\chi_E(\left[5 \,\middle\|\, 2\ 4\right]) = -22 \cdot 8 = -176\ . \tag{C.27}$$

We need one more ingredient in order to compute the Hodge numbers of our Calabi-Yaus:

Theorem C.30 (Lefshetz Hyperplane Theorem)**.** *Let X be a compact complex manifold and $Y \subset X$ a smooth hypersurface such that $\mathcal{O}(Y)$ is a positive line bundle. (A line bundle is positive if there is a metric such that its curvature is a positive form, or equivalently, if its first Chern class has a positive representative.)*

The map $i^* : H^q(X) \to H^q(Y)$ is an isomorphism for $q < \dim_{\mathbb{C}} Y$ and injective for $q = \dim_{\mathbb{C}} Y$.

Proof. Omitted, see [48, Ch. 1.2] or [58, Prop. 5.2.6]. □

Note. In this context, *ample* line bundles are sometimes mentioned. Ampleness is another property line bundles can have, see [58, Def. 2.3.28]. The *Kodaira embedding theorem* tells us that over a compact Kähler manifold, line bundles are ample if and only if they are positive [58, Prop. 5.3.1].

We can apply the Lefshetz hyperplane theorem because $\mathcal{O}(1)$ in $\mathbb{P}^n$ is a positive bundle, and its powers are as well. In our case, we can apply this theorem several times until we arrive at our Calabi-Yau, and see that $b^{1,1}$ doesn't change along the way. Thus $b^{1,1} = 1$ for all our complete intersections and $b^{2,1}$ can be computed from $b^{1,1}$ and χ_E. Summarizing:

Label	$b^{1,1}$	$b^{2,1}$	χ_E
$\left[4 \,\middle\|\, 5\right]$	1	101	-200
$\left[5 \,\middle\|\, 2\ 4\right]$	1	89	-176
$\left[5 \,\middle\|\, 3\ 3\right]$	1	73	-144
$\left[6 \,\middle\|\, 2\ 2\ 3\right]$	1	73	-144
$\left[7 \,\middle\|\, 2\ 2\ 2\ 2\right]$	1	65	-128

Taken from [59, Table 1.1].

C.2 Constructing Calabi-Yaus

C.2.1 Products of Projective Spaces

We want to consider complete intersections in the embedding space $X = \mathbb{P}_1^{n_1} \times \cdots \times \mathbb{P}_m^{n_m}$. A hypersurface in X is given as the zero locus of a polynomial which can have different degrees q^r of homogeneity in the coordinates of the individual $\mathbb{P}_r^{n_r}$. Such a polynomial is a section of the line bundle $\bigotimes_r \mathcal{O}_r(q^r)$.

A complete intersection is the intersection of k such hypersurfaces, in other words the zero locus of a section of

$$\bigoplus_{a=1}^{k} \left(\bigotimes_{r=1}^{m} \mathcal{O}_r(q_a^r) \right)\ . \tag{C.28}$$

Definition C.31 (Configuration Matrix)**.** The degrees of homogeneity of the defining polynomials are written as the *configuration matrix* of the respective complete intersection. It looks like this:

$$\left[\boldsymbol{n} \,\middle\|\, q\right] = \left[\begin{matrix} n_1 \\ \vdots \\ n_m \end{matrix} \,\middle\|\, \begin{matrix} q_1^1 & \cdots & q_k^1 \\ \vdots & \ddots & \vdots \\ q_1^m & \cdots & q_k^m \end{matrix}\right]\ . \tag{C.29}$$

Such a configuration matrix stands for the set of all complete intersections with the given degrees.

This is obviously a generalization of definition C.24.

```
> q[1][1] := 2; q[1][2] := 2; q[1][3] := 0; q[2][1] := 1; q[2][2] := 1; q[2][3] := 2
> Coefficient[
      Sum[-1/3 * q[a][r]*q[a][s]*q[a][t] * J[r]*J[s]*J[t],
        {a,1,2}, {r,1,3}, {s,1,3}, {t,1,3}] *
      Product[Sum[q[a][p]*J[p], {p,1,3}], {a,1,2}],
    J[1]^2 J[2]^2 J[3]]
< -96
```

Listing C.1: Calculating the Euler characteristic in example C.36 using Mathematica.

Example C.32. A complete intersection with configuration matrix

$$\left[\begin{array}{c||ccc} 3 & 3 & 0 & 1 \\ 3 & 0 & 3 & 1 \end{array}\right] \tag{C.30}$$

is the intersection of three hypersurfaces in $\mathbb{P}^3_1 \times \mathbb{P}^3_2$. The first is given by a polynomial of degree 3 in the coordinates x of $\mathbb{P}^3_1$ only: $f^{abc}x_a x_b x_c = 0$. Similarly, the second hypersurface is given by a polynomial of degree 3 in the coordinates y of $\mathbb{P}^3_2$ only: $g^{\alpha\beta\gamma}y_\alpha y_\beta y_\gamma = 0$. The polynomial defining the third is of degree 1 in both coordinate sets: $h^{a\alpha}x_a y_\alpha = 0$.

Theorem C.33 (Chern Classes)**.** *Let J_r be the hyperplane classes of the individual spaces $\mathbb{P}^{n_r}_r$. Then*

$$\mathrm{c}\left[\boldsymbol{n}\,\middle\|\,\underline{q}\right] = \frac{\prod_r (1+J_r)^{n_r+1}}{\prod_a (1+\sum_r q^r_a J_r)} . \tag{C.31}$$

The first Chern classes are $\mathrm{c}_1\left[\boldsymbol{n}\,\middle\|\,\underline{q}\right] = \sum_r \left(n_r + 1 - \sum_a q^r_a\right) J_r$.

We see that the complete intersection is Calabi-Yau if

$$\sum_a q^r_a = n_r + 1 \tag{C.32}$$

for all rows r. Sometimes we are interested in cases which have an inequality "$<$" instead of the "$=$" because such subvarieties have positive hyperplane bundles, similar to projective space.

Definition C.34. A configuration with $\sum_a q^r_a < n_r + 1$ for all r is called *ample*.
A configuration with $\sum_a q^r_a \leq n_r + 1$ for all r, with a strict inequality for at least one r, is called *almost ample*.

Theorem C.35 (Euler Characteristic)**.** *With the methods detailed above, we calculate for a Calabi-Yau complete intersection:*

$$\chi_E\left[\boldsymbol{n}\,\middle\|\,\underline{q}\right] = \left[\sum_{r,s,t} \frac{1}{3}\left(\delta^{rst}(n_r+1) - \sum_a q^r_a q^s_a q^t_a\right) J_r J_s J_t \cdot \bigwedge_a \left(\sum_{p=1}^m q^p_a J_p\right)\right]_{top} , \tag{C.33}$$

"top" meaning the coefficient of $\prod_r J_r^{n_r}$.

Example C.36. We'll consider a complete intersection

$$V \in \left[\begin{array}{c||cc} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 2 \end{array}\right] . \tag{C.34}$$

It is tedious but straightforward to calculate $\chi_E = -96$ from (C.33), see listing C.1.

In order to calculate $b^{1,1}$, we want to apply the Lefshetz hyperplane theorem again. This is a bit tricky here because of the requirement that $\mathcal{O}(Y)$ is positive. As described in [59, Ch. 2.4.1], it works for example when the configuration is *favourable* (which is a certain condition on the location of zeroes in the matrix).

The configuration we are investigating here is favourable, we have to start with the first column and calculate the Hodge numbers of $\tilde{V} \in \left[\begin{matrix}2\\2\end{matrix}\middle\|\begin{matrix}2\\2\end{matrix}\right]$. With the *Künneth formula*

$$b^q(X \times Y) = \sum_{p=0}^{q} b^{q-p}(X)b^p(Y) \tag{C.35}$$

we get $b^{2q+1}(\mathbb{P}^2 \times \mathbb{P}^2) = 0$ and $b^{2q}(\mathbb{P}^2 \times \mathbb{P}^2) = (1,2,3,2,1)$. Because of the hyperplane theorem, $b^q(\tilde{V}) = b^q(\mathbb{P}^2 \times \mathbb{P}^2)$ for $q < 3$. This suffices to calculate $b^2(\tilde{V} \times \mathbb{P}^1) = 3$, by the hyperplane theorem also $b^2(V) = b^{1,1}(V) = 3$.

Summarizing: $\chi_E = -96$, $b^{1,1} = 3$ and $b^{2,1} = b^{1,1} - \frac{1}{2}\chi_E = 51$.

As seen in example C.32, one configuration matrix stands for a large class of varieties. An important results states that most of them are non-singular and deduces that they are all essentially identical:

Theorem C.37 (Bertini)**.** *A generic element of a linear system is smooth (away from the base locus[3] of the system).*

Since the set of constraints leading to non-singular Calabi-Yaus is connected, all smooth manifolds belonging to one configuration can be smoothly deformed into each other.

Proof. This can be proven by counting the number of constraints and of degrees of freedom.

Take for example the quintic in $\mathbb{P}^4$ given by $F(z) = f^{abcde}z_a z_b z_c z_d z_e = 0$. As described in [59, Ch. 2.2.1], the parameter space $\mathfrak{M}$ of coefficients f^{abcde} is 101-dimensional. Let $z \in \mathbb{P}^4$ be a single point. The manifold described by F is singular in z if $F(z) = 0$ and $\mathrm{d}F(z) = 0$, actually only 5 of those 6 conditions are independent. Therefore the parameter space $\mathfrak{M}^\sharp_z$ of coefficients leading to manifolds with a singularity in z has $\dim \mathfrak{M}^\sharp_z = \dim \mathfrak{M} - 5$.

Now the dimension of the space $\mathfrak{M}^\sharp$ of parameters leading to manifolds with a singularity somewhere is $\dim \mathfrak{M}^\sharp = \dim \mathfrak{M}^\sharp_z + \dim \mathbb{P}^4 = \dim \mathfrak{M} - 1$, one dimension less than the total parameter space. □

Note. A large part of chapter 2 of [59] is devoted to classifying CY complete intersections in product projective spaces. Following theorem C.37, we only have to classify configuration matrices (and still different configuration matrices can lead to isomorphic CY spaces).

For the classification we can first exclude "boring" cases like a block-diagonal configuration matrix, or one with a column that only contains a single "1" somewhere. Going on, [59] introduce a diagrammatic notation for configuration matrices: Draw a hollow circle with $n_r + 1$ legs for each row r and a dot for each column. Connect then q^r_a of the legs of the circle r to the dot a.

With the help of some rules for simplifying such diagrams, they finally arrive at the result: There are some $97,000$ different minimal configurations with over $6,000$ topologically distinct CY 3-folds.

C.2.2 Blowing Up

An important technique for constructing new complex manifolds is *blowing up*. Blowing up an n-dimensional manifold in a point essentially means to replace that point with $\mathbb{P}^{n-1}$.

Definition C.38 (Blow-Up)**.** Let X be an n-dimensional complex manifold and $x \in X$. There exists a complex manifold $\hat{X}$, called the *blow-up of X in x*, together with a holomorphic map $\sigma : \hat{X} \to X$ such that

i) the *exceptional divisor* $E = \sigma^{-1}(x)$ is isomorphic to $\mathbb{P}^{n-1}$, and

ii) $\hat{X} \backslash E$ and $X \backslash \{x\}$ are isomorphic via σ.

[3] In our case that is the locus where all possible choices of the defining polynomial vanish. Note that for simple cases like the quintic in $\mathbb{P}^4$, the base locus is empty.

Example C.39. Consider $\mathcal{O}_{\mathbb{P}^{n-1}}(-1) = \{(\ell, x) \in \mathbb{P}^{n-1} \times \mathbb{C}^n : x \in \ell\}$ as a complex manifold together with the projection $\sigma : \mathcal{O}(-1) \to \mathbb{C}^n$ (i.e. $\sigma(\ell, x) = x$). This is the blow-up of $\mathbb{C}^n$ in 0.

Note that we can also write this as

$$\widehat{\mathbb{C}^n} = \{(x, z) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : x_i z_j = x_j z_i \text{ for all } 0 \leq i, j < n\} \tag{C.36}$$

where z_i are the homogeneous coordinates of $\mathbb{P}^{n-1}$ and x_i the coordinates of $\mathbb{C}^n$.

Note. This example makes it clear how we can construct the blow-up in general: Blowing up is a local operation and the complex manifold X always looks like $\mathbb{C}^n$ in a neighborhood of x.

Note. More generally, one can also define the blow-up along a linear subspace $Y \subset X$. Let for example $X = \mathbb{C}^n$ (with coordinates $x_0, \ldots, x_{n-1}$ like before) and $Y = \mathbb{C}^k = \{x \in \mathbb{C}^n : x_k = \cdots = x_{n-1} = 0\}$. Then the *blow-up of X along Y* is

$$\{(x, z) \in \mathbb{C}^n \times \mathbb{P}^{n-k-1} : x_i z_j = x_j z_i \text{ for all } 0 \leq i, j < (n-k)\} . \tag{C.37}$$

Theorem C.40 (Canonical Bundle of the Blow-Up)**.** *Let $\hat{X}$ together with $\sigma : \hat{X} \to X$ be the blow-up of the n-dimensional complex manifold X in $x \in X$. Then*

$$K_{\hat{X}} \cong \sigma^* K_X \otimes \mathcal{O}_{\hat{X}}((n-1)E) . \tag{C.38}$$

This implies especially

$$\mathcal{O}(E)\big|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1) . \tag{C.39}$$

Proof. The proof of the first part can be found in [58, Prop. 2.5.5]. The general idea is to prove the claim locally, i.e. for the case $X = \mathbb{C}^n$ with $K_X = \mathcal{O}$. Then we compare the cocycles of $K_{\hat{X}} = K_{\mathcal{O}(-1)}$ and of $\mathcal{O}_{\hat{X}}(E)$.

(C.39) follows because

$$\mathcal{O}_{\mathbb{P}^{n-1}}(-n) = K_{\mathbb{P}^{n-1}} = \left(K_{\hat{X}} \otimes \mathcal{O}(E)\right)\big|_E = \left(\sigma^* K_X \otimes \mathcal{O}(nE)\right)\big|_E$$

(we used the adjunction formula (B.51) and the first part of the theorem). $\sigma^* K_X|_E$ is zero because $i^* \sigma^* K_X = (\sigma \circ i)^* K_X = \mathcal{O}$. Because $\mathrm{Pic}(\mathbb{P}^{n-1}) \cong \mathbb{Z}$, we conclude (C.39). ☐

Example C.41 (Blow-Up of $\mathbb{P}^n$)**.** Let's construct the blow-up of $\mathbb{P}^n$ in $[1 : 0 : \cdots : 0]$. Since blowing-up is a local operation, we can just blow up in the chart $U_0 = \{[1 : x_1 : \cdots : x_n]\} \cong \mathbb{C}^n$. We know that we only have to take the product with $\mathbb{P}^{n-1} = \{[z_1 : \cdots : z_n]\}$ and demand $x_i z_j = x_j z_i$ for all $1 \leq i, j \leq n$.

The complete result is therefore

$$\widehat{\mathbb{P}^n} = \left\{([x_0 : \cdots : x_n], [z_1 : \cdots : z_n]) \in \mathbb{P}^n \times \mathbb{P}^{n-1} : x_i z_j = x_j z_i \text{ for all } 1 \leq i, j \leq n\right\} . \tag{C.40}$$

There is a more practical way of constructing the blow-up which will also work in more general settings. In order to understand it, we need to adopt a different, more abstract way of thinking about projective space: We can describe a large class of varieties by just giving the following data:

- The quasi-homogeneous coordinates $x_1 \ldots x_m$ we will use.
- Some *scaling relations* of those coordinates, for example we could want to identify $[x_1 : x_2 : x_3]$ with $[\Lambda x_1 : \Lambda^2 x_2 : x_3]$. Such scaling relations are usually written down in a table like this:

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \hline 1 & 2 & 0 \end{array} .$$

- The *Stanley-Reisner ideal* or *SR ideal* is the ideal in the ring of monomials over the coordinates containing those combinations of coordinates that are not allowed to be zero at the same time.

Example C.42 (Blow-Up of $\mathbb{P}^2$)**.** For definiteness' sake we consider $\mathbb{P}^2$, we want to blow it up in $[1:0:0]$. We describe $\mathbb{P}^2$ using three coordinates, say x, y and z, with the simple scaling relation $\begin{array}{ccc} x & y & z \\ \hline 1 & 1 & 1 \end{array}$ and the SR ideal $\langle xyz \rangle$.

We get the blow-up by introducing a new coordinate λ and scaling the coordinates y and z with it. More precisely, we claim that the blow-up is:

$$\begin{array}{cccc} x & y & z & \lambda \\ \hline 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array} \tag{C.41}$$

with an SR ideal of $\langle xyz, yz, x\lambda \rangle = \langle yz, x\lambda \rangle$. The blow-up map σ is given by $\sigma(x,y,z,\lambda) = [x : \lambda y : \lambda z]$.

Note. We can e.g. subtract the second row from the first in (C.41) and use the resulting relation instead of the second, say. What we get are equivalent scaling relations: $\begin{array}{cccc} x & y & z & \lambda \\ \hline 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}$.

Proof that this describes the blow-up of $\mathbb{P}^2$*.* First we easily check that σ is well-defined, i.e. $\sigma(x,y,z,\lambda) = \sigma(\Lambda x, \Lambda y, \Lambda z, \lambda) = \sigma(x, \Lambda y, \Lambda z, \Lambda^{-1}\lambda)$.

Then we calculate the exceptional divisor: $\sigma^{-1}[1:0:0] = \{(1,y,z,0)\} = \mathbb{P}^1$ because y and z can't be zero at the same time.

Finally we see that any other $[x:y:z] \in \mathbb{P}^2$ has exactly one preimage, namely the class of $(x,y,z,1)$. $\square$

Note. This kind of varieties are actually *toric varieties* [66, Thm. 5.1.11].

Toric varieties are in a sense a generalization of all the kinds of manifolds we are considering in this section. For example, products of weighted projective spaces are toric varieties. We refer to [66], to the appendix of [10] and also to subsection 2.2.4 for more on toric varieties.

C.2.3 Singularities

We consider the action of some group G on a variety Y.

Definition C.43 (Some Group Theory)**.** An *orbit* of Y under this group action is an equivalence class where $y \sim y'$ iff $y = g.y'$ for some $g \in G$.

The group action is *free* if, for all $y \in Y$, $g.y = y$ implies $g = e$.

The *order* of an element $g \in G$ is the smallest positive integer m such that $g^m = e$, it divides the order of the group $|G|$.

This is interesting because all previously constructed CY manifolds were simply connected and hence not suitable for physics applications. The situation changes if we go from Y to $M = Y/G$ (i.e. the set of orbits with the quotient topology, which is a complex manifold again):

Theorem C.44. *A necessary condition for M to be a smooth CY manifold again (if Y was CY) is that the action of G is holomorphic and free.*

Then G becomes the fundamental group of M (if Y was simply connected) and

$$\chi_E(M) = \frac{\chi_E(Y)}{|G|} \ . \tag{C.42}$$

In reality however it is difficult to construct models where the group action is free. If it is not free, each group element $g \in G$ can fix a set $S_g = \{y \in Y : g.y = y\}$ of arbitrary codimension less than n. We have to expect that M has singularities in those fixed sets, M is not a manifold but a so-called *orbifold*.

In general, sets S_g of codimension 1 can not be singular. For a CY 3-fold we can have singularities of dimension 0 or 1. We will concentrate on point-like singularities of dimension 0. Those will be resolved by replacing the singular points by an exceptional set, a simple argument shows that it should be of complex dimension 1 or 2.

We already know of a process which replaces a point in a CY 3-fold by a complex set of dimension 2, namely the blow-up. In the case where the space M that we want to blow up has a singularity, we have to modify (C.38) though. If locally $M = \mathbb{C}^n/\mathbb{Z}_k$, the blow-up will be $\mathcal{O}_{\mathbb{P}^{n-1}}(-k)$ instead of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and thus

$$K_{\hat{M}} \cong \sigma^* K_M \otimes \mathcal{O}_{\hat{X}}\left((n-k)E\right) \ . \tag{C.43}$$

The blow-up of a singularity of a type other than $\mathbb{Z}_3$ will not be CY any more if M was CY. One has to do more work and consider so-called *toric resolutions*, see [59, Ch. 4.3].

Replacing the singularity by a complex set of dimension 1 is called *small resolution*. The exceptional set here consists of a chain of $\mathbb{P}^1$'s intersecting each other as described by the Dynkin diagram of some Lie algebra [59, Ch. 4.4].

Lemma C.45. *We take $M = Y/G$ like above. Let $\{S_i\}$ be the set of irreducible components of the fixed point set S and $g_i \in G$ the corresponding group elements. Finally, let $\hat{M}$ be a resolution with exceptional set E. Then*

$$\chi_E(\hat{M}) = \frac{\chi_E(Y)}{|G|} - \sum_i \frac{\chi_E(F_i)}{|g_i|} + \chi_E(E) \ , \tag{C.44}$$

where $|g_i|$ is the order of $g_i \in G$ [59, Ch. 4.5].

C.2.4 Weighted Projective Spaces

Definition C.46. A *weighted projective space* $\mathbb{P}^n_{(w_0:\cdots:w_n)}$ is a space defined by the scaling relation

$$\begin{array}{cccc} z_0 & z_1 & \cdots & z_n \\ \hline w_0 & w_1 & \cdots & w_n \end{array} \tag{C.45}$$

(notation like in subsection C.2.2). The SR ideal is $\langle z_0 \cdots z_n \rangle$ like in ordinary projective space.

We write the weight in the form $(w_0 : \cdots : w_n)$ like an element of $\mathbb{P}^n$ to visualize the fact that $(w_0 : \cdots : w_n)$ and $(\Lambda w_0 : \cdots : \Lambda w_n)$ obviously describe the same space. In other words, if all the weights have a common divisor, we can simplify the system of weights.

In fact, we can simplify the system already if a subset of n of the weights has a common divisor (greater than one):

Definition C.47. We'll call the greatest common divisors

$$d_i = \gcd(w_0 \dots \hat{w}_i \dots w_n) \ . \tag{C.46}$$

The system of weights is called *well-formed* if all of the d_i are one.

If some d_i are greater than one, we can simplify. The best we can hope to achieve is that we can divide w_i by the least common multiple of all those d_j that divide w_i (i.e. those with $j \neq i$). Indeed this is possible:

Theorem C.48 (Delorme)**.** *Let $m_i = \mathrm{lcm}(d_0 \dots \hat{d}_i \dots d_n)$. Then*

$$\mathbb{P}^n_{(w_0:\cdots:w_n)} \cong \mathbb{P}^n_{(w_0/m_0:\cdots:w_n/m_n)} \tag{C.47}$$

and the new weight system is well-formed.

Example C.49. Consider $\mathbb{P}^2_{(1:2:2)}$. The greatest common divisors are $d = (2,1,1)$, therefore $\mathbb{P}^2_{(1:2:2)} = \mathbb{P}^2$.

Example C.50. $\mathbb{P}^4_{(2:3:6:12:18)} = \mathbb{P}^4_{(1:1:1:2:2)}$ because

w	2	3	6	12	18
d	3	2	1	1	1
m	2	3	6	6	6
w/m	1	1	1	2	2

.

Theorem C.51 (More Properties of Weighted $\mathbb{P}^n$)**.**

i) A weighted projective space is a projective manifold and thus Kähler.

ii) The coordinates z_i are sections of $\mathcal{O}(w_i)$ and the Euler sequence (C.8) *is modified to*

$$0 \to \mathcal{O} \to \bigoplus_{i=0}^{n} \mathcal{O}(w_i) \to T\left(\mathbb{P}^n_{(w_0:\cdots:w_n)}\right)^+ \to 0 \; . \tag{C.48}$$

iii) This modifies the formulae for the Chern classes etc. slightly, e.g. (more in [59, 5.1])

$$c\left(\mathbb{P}^n_{(w_0:\cdots:w_n)}\right) = \prod_{i=0}^{n}(1 + w_i J) \tag{C.49}$$

and $c_1(V) = \sum_{i=0}^{n} w_i - \sum_{a=1}^{k} q_a$ for a complete intersection V, where the degree of the polynomial F_a is equal to q_a if it is a section of $\mathcal{O}(q_a)$.

iv) There are singularities: Consider for example $\mathbb{P}^3_{(1:1:1:2)}$, a neighborhood of $(0:0:0:1)$ looks like $\mathbb{C}^3/\mathbb{Z}_2$.

In general, if $w_i > 1$, then a neighborhood of $(0:\cdots:1:\cdots:0)$ will look like $\mathbb{C}^n/\mathbb{Z}_{w_i}$. Also, if $\gcd(w_i, w_j) > 1$ then the subspace of dimension one described by the coordinates z_i and z_j will be singular, and so on.

Note that in a 3-dimensional weighted projective space we can only have singularities of dimension 0 or 1, because the gcd *of three weights is already one by definition.*

Complete intersections in (products of) weighted projective spaces have to deal with singularities of two types: They can inherit singularities from the embedding space, but they can also acquire singularities because the base locus in Bertini's theorem C.37 might be non-trivial. However:

Theorem C.52 (Quoted from [59, Thm. 5.4])**.** *For a quasismooth[4] weighted complete intersection 2- or 3-fold X in a product of weighted projective spaces, the minimal desingularization $\hat{X}$ of X is a smooth CY manifold as long as a condition analogous to* (C.32) *is satisfied.*

[4] Meaning that there are only inherited, no acquired singularities.

Appendix

Appendix D

Code

D.1 Calculating Blow-Ups

It is easy to write a short program calculating the blow-up of a toric variety as described in subsection 2.2.4. The function calculating the blow-up in such an API might look like this:

Calculating blow-ups in Java.

```java
public ToricVariety blowUp(String coordinateName, Set<Coordinate> locus) {
    Coordinate newCoordinate = new Coordinate(coordinateName);

    ScalingRelation newRelation = ScalingRelation.EMPTY.append(newCoordinate, -1);
    for (Coordinate coord : coordinates) {
        if (locus.contains(coord)) {
            newRelation = newRelation.append(coord, 1);
        } else {
            newRelation = newRelation.append(coord, 0);
        }
    }

    ToricVariety t = this.appendCoordinate(newCoordinate).appendRelation(newRelation);

    MonomialIdeal srIdeal = t.srIdeal;
    t = t.addToSRIdeal(new Monomial(locus));
    for (Monomial m : srIdeal.generators) {
        if (containsAny(locus, m.coords)) {
            Set<Coordinate> newGenerator = new HashSet<>(generator.coords);
            for (Coordinate coord : locus)
                newGenerator.remove(coord);
            newGenerator.add(newCoordinate);
            t = t.addToSRIdeal(newGenerator);
        }
    }
    return t;
}
```

This code was mainly meant to illustrate the algorithm described in subsection 2.2.4. But such simple means are already sufficient to calculate the series of blow-ups performed in section 6.2, as demonstrated here:

```java
// ...

ToricVariety Y4 = new ToricVariety(coordSet, relations, srIdeal);
System.out.println(Y4);

ToricVariety T1 = Y4.blowUp("v1", new Monomial(z5, X, Y).coords);
Coordinate v1 = T1.coordinateNamed("v1");

ToricVariety T2 = T1.blowUp("v2", new Monomial(v1, X, Y).coords);
```

```
Coordinate v2 = T2.coordinateNamed("v2");

ToricVariety T3 = T2.blowUp("v4", new Monomial(v1, Y).coords);
ToricVariety T4 = T3.blowUp("l", new Monomial(X, Y).coords);
ToricVariety result = T4.blowUp("v3", new Monomial(v2, Y).coords);

System.out.println(result);
```

In subsection D.3.2, we will show how to do the same in Sage.

D.2 Calculations in the SU(2) Example

One computer algebra system that can handle toric varieties quite well is *Sage*. All $\mathbb{P}^n$ are predefined toric varieties. If we want to define the blow-up in a point, we have to analyze the fan used to represent $\mathbb{P}^4$ and add the new ray $-(-1,-1,-1,-1)$ manually:

```
> P4 = toric_varieties.P(4)
> P4.fan().rays() # Sage has no built-in functions for blowing up,
                  # we have to add the ray to the fan manually
< N( 1,  0,  0,  0),
< N( 0,  1,  0,  0),
< N( 0,  0,  1,  0),
< N( 0,  0,  0,  1),
< N(-1, -1, -1, -1)
< in 4-d lattice N

> T4 = P4.resolve(new_rays=[(1,1,1,1)])

> T4.gens() # coordinates
< (z0, z1, z2, z3, z4, z5)

> T4.rational_class_group()._projection_matrix
     # scaling relations = mapping Div -> Div/~
< [ 1  1  1  1  0 -1]
< [ 0  0  0  0  1  1]

> T4.Stanley_Reisner_ideal()
< Ideal (z4*z5, z0*z1*z2*z3) of Multivariate Polynomial Ring
     in z0, z1, z2, z3, z4, z5 over Rational Field
```

We can easily read off the volume forms of $\mathbb{P}^4$ and T_4 (compare the discussion around (5.50)) and perform calculations in the cohomology of T_4:

```
> P4.volume_class()
< [z4^4]
> T4.volume_class()
< [-z5^4]

> z=T4.divisor(0).cohomology_class()
> w=T4.divisor(5).cohomology_class()
> z4=T4.divisor(4).cohomology_class()
> z4 == z+w
< True

> T4.integrate(z^4); T4.integrate(z^3 * w); T4.integrate(z^2 * w^2);
     T4.integrate(z * w^3); T4.integrate(w^4)
< 0; 1; -1; 1; -1
```

Also, we can check ampleness of the divisor B_3

```
> B3 = T4.divisor([3,0,0,0,0,1])
> B3.is_ample()
< True
```

and confirm our results about the Chern classes, (5.55).

```
> P4.c(1)
< [5*z4]
> T4.c(1)
< [5*z4 - 3*z5]
> B3.ch().part_of_degree(1) # 1st chern character == 1st chern class
< [3*z4 - 2*z5]

> T4.c(1) - B3.ch().part_of_degree(1) == 2*z+w
< True
```

D.3 The SU(5) Model

D.3.1 Calculations Concerning T_4

The toric variety T_4 was defined in (6.1). First, we need to enter this variety in Sage. With a short calculation, we find a set of rays satisfying the appropriate linear relations. Also, we write down a set of maximal cones in such a way that the *primitive collections* of the fan correspond to the generators of the SR ideal:

```
> fan = Fan( # Write down a fan for T_4
>     cones=[(0, 1, 3, 4), (0, 1, 3, 5), (0, 1, 4, 5), (0, 2, 3, 4), (0, 2, 3, 5),
            (0, 2, 4, 5), (1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 4, 5)],
>     rays=[(1,0,0,0),(0,1,0,0),(-1,-1,-1,1),(0,0,1,0),(0,0,0,1),(0,0,-1,-1)])
> fan.primitive_collections()
< [frozenset([0, 1, 2]), frozenset([3, 4, 5])]

> T4 = ToricVariety(fan, coordinate_names="z1␣z2␣z3␣z4␣z5␣z6")
> T4.rational_class_group()._projection_matrix
> T4.Stanley_Reisner_ideal()
< [ 1  1  1  0 -2 -1]
< [ 0  0  0  1  1  1]
< Ideal (z1*z2*z3, z4*z5*z6) of Multivariate Polynomial Ring in
          z1, z2, z3, z4, z5, z6 over Rational Field

> T4.divisor([5, 0, 0, 0, 2, 0]).is_ample()
< True
```

We also checked that B_3 is an ample divisor of T_4.

Let us now define basic quantities in homology, namely the $[z_i]$, the $[\mathfrak{b}_i]$ and $[B_3]$:

```
> z1 = T4.divisor(0).cohomology_class()
> z2 = z1; z3 = z1
> z4 = T4.divisor(3).cohomology_class()
> z5 = T4.divisor(4).cohomology_class()
> z6 = T4.divisor(5).cohomology_class()

> b2 = 4*z1 + z5
> b3 = 3*z1 + z5
> b4 = 2*z1 + z5
> b5 = z1 + z5
> B3 = 5*z1 + 2*z5
```

This can be used to perform the calculation of $\mathrm{U}(1)_X$ chiral indices in section 6.5, equations (6.56), (6.57), (6.59), (6.60) and (6.62):

```
> -2*T4.integrate(z2*z4*z5*(z1-8*z6)) # Hd ~ z2 z4 z5
> -2*T4.integrate(B3*(b3+b4)*z5*(z1-8*z6)) # H ~ B3 (b3+b4) z5
< 3*T4.integrate(B3*b3*z5*(z1-8*z6)) # 5M
< 1*T4.integrate(B3*b5*z5*(z1-8*z6)) # 10M
< 5*T4.integrate(B3*b2*b3*(z1-8*z6)) # 1
< -2
< 0
< -9
< 9
< -1095
```

And also their generalizations in section 6.6, equation (6.65):

```
> a = var('a'); b = var('b')
> a * (-2*T4.integrate(z1*z4*z5*z1)) + b * (-2*T4.integrate(z1*z4*z5*z6))
> a * (-2*T4.integrate((B3*(b3+b4) - z1*z4)*z5*z1)) +
        b * (-2*T4.integrate((B3*(b3+b4) - z1*z4)*z5*z6))
> a * (3*T4.integrate(B3*b3*z5*z1)) + b * (3*T4.integrate(B3*b3*z5*z6))
> a * (T4.integrate(B3*b5*z5*z1)) + b * (T4.integrate(B3*b5*z5*z6))
> a * (5*T4.integrate(B3*b2*b3*z1)) + b * (5*T4.integrate(B3*b2*b3*z6))
< -2*a
< -14*a - 2*b
< 15*a + 3*b
< a - b
< 65*a + 145*b

> # ---------- CASE WHERE C_Hd = {z1, z3} ----------
> a * (-2*T4.integrate(z1*z3*z5*z1)) + b * (-2*T4.integrate(z1*z3*z5*z6))
< -2*b
```

(in the last part, we calculated (6.86)).

Finally, we can use (6.72) to check the calculations (6.73), (6.75) and (6.76) that will also be done in subsection D.3.2:

```
> Kbar = z1+z5; S = z5
> -T4.integrate(B3*S*S*Kbar) # chi_lambda(5H)
> 2 * T4.integrate(B3*b3*z5*Kbar) # chi_lambda(5M)
> T4.integrate(B3*b5*z5*(-6*Kbar+5*S)) # chi_lambda(10M)
< 2
< 2
< -4
```

D.3.2 Calculations Concerning T_6

Let us first write down T_6 (see (6.14)) in Sage. To do so, we automatically generate a list of maximal cones in such a way that the primitive collections correspond to the generators of the SR ideal:

```
> rays = [
>     ( 1, 0, 0, 0, 0, 0),
>     ( 0, 1, 0, 0, 0, 0),
>     (-1,-1,-1, 1, 0, 0),
>     ( 0, 0, 1, 0, 0, 0),
>     ( 0, 0, 0, 1, 0, 0),
>     ( 0, 0,-1,-1,-2,-3),
>     ( 0, 0, 0, 0, 1, 0),
>     ( 0, 0, 0, 0, 0, 1),
>     ( 0, 0, 0, 0,-2,-3)
> ]

> def is_subset(list1, list2):
>     return all( [(elem in list2) for elem in list1] )

> possible_maximal_cones = Subsets(range(9), 6)
> maximal_cones = [ cone.list() for cone in possible_maximal_cones if not
>                   (is_subset([0,1,2], cone) or is_subset([3,4,5], cone)
                     or is_subset([6,7,8], cone)) ]
> fan = Fan(maximal_cones, rays)
> fan.primitive_collections()
< [frozenset([0, 1, 2]), frozenset([3, 4, 5]), frozenset([8, 6, 7])]

> names="z1 z2 z3 z4 z5 z6 x y z v1 v2 v4 l v3"

> T6 = ToricVariety(fan, coordinate_names=names)
> T6.rational_class_group()._projection_matrix
> T6.Stanley_Reisner_ideal()
< [ 1  1  1  0 -2 -1  0  0  1]
< [ 0  0  0  1  1  1  0  0 -1]
< [ 0  0  0  0  0  0  2  3  1]
```

```
<    Ideal (z1*z2*z3, z4*z5*z6, x*y*z) of Multivariate Polynomial Ring
                         in z1, z2, z3, z4, z5, z6, x, y, z over Rational Field
```

Now we can simply follow the blow-up route detailed in section 6.2 and get $\tilde{T}_6$ (see (6.17)):

```
>    def add2(v1, v2):
>        return map(operator.add, v1, v2)
>    def add3(v1, v2, v3):
>        return add2(v1, add2(v2, v3))

>    ray_v1 = add3(rays[4], rays[6], rays[7])
>    T6_1 = T6.resolve(new_rays=[ray_v1], coordinate_names=names)
>    ray_v2 = add3(ray_v1, rays[6], rays[7])
>    T6_2 = T6_1.resolve(new_rays=[ray_v2], coordinate_names=names)
>    ray_v4 = add2(ray_v1, rays[7])
>    T6_3 = T6_2.resolve(new_rays=[ray_v4], coordinate_names=names)
>    ray_l = add2(rays[6], rays[7])
>    T6_4 = T6_3.resolve(new_rays=[ray_l], coordinate_names=names)
>    ray_v3 = add2(ray_v2, rays[7])
>    T6_tilde = T6_4.resolve(new_rays=[ray_v3], coordinate_names=names)

>    T6_tilde.Stanley_Reisner_ideal()
<    Ideal (z5*v2, z5*l, z5*v3, x*y, x*v4, x*v3, y*v1, z*v1, v1*l, v1*v3, y*v2,
             z*v4, v4*l, z*v2, z*l, z*v3, z1*z2*z3, z4*z5*z6, z4*z6*v1, z4*z6*v4,
             z4*z6*v2, z4*z6*v3) of Multivariate Polynomial Ring in
             z1, z2, z3, z4, z5, z6, x, y, z, v1, v2, v4, l, v3 over Rational Field
```

Next, let us quickly define all relevant cohomology classes. B_3 and eq_w are the classes of the two equations in (6.16), the rest should be self-explanatory.

```
>    z1 = T6_tilde.divisor(0).cohomology_class()
>    z2 = T6_tilde.divisor(1).cohomology_class()
>    z3 = T6_tilde.divisor(2).cohomology_class()
     [...]
>    ell = T6_tilde.divisor(12).cohomology_class()
>    v3 = T6_tilde.divisor(13).cohomology_class()

>    eq_w = v3 + v4 + ell + 2*y
>    b3 = 2*z6 + 3*z1

>    Kbar = z1 + z5 + v1 + v2 + v3 + v4
>    S = z5 + v1 + v2 + v3 + v4
>    b5 = Kbar
>    b4 = 2*Kbar - S
>    b3 = 3*Kbar - 2*S
>    b2 = 4*Kbar - 3*S
```

The G_4^λ and G_4^X fluxes are defined as:

```
>    Glambda = 5 * v2 * v4 + 2 * v1 * Kbar - v2 * Kbar + v3 * Kbar - 2 * v4 * Kbar
>    GX = (z1 - 8*z6) * (-5*(ell - z - Kbar) - 2*v1 - 4*v2 - 6*v3 - 3*v4)
```

We will go through all the matter surfaces and calculate some of the χ_λ chiral indices. Note that some of them were also calculated in subsection D.3.1, but others can not as easily be reduced to an integral over T_4. For each matter curve, we will in the first line compute the intersection products of G_4^λ with the matter surfaces corresponding to the roots of the respective representation. All of these should be zero because G_4^λ flux is orthogonal to the Cartan fluxes, (6.71), such that all components of the representation have the same chiral index. In the second line we then compute that chiral index.

```
>    # 5M surface
>    # roots (just to check)
>    [T6_tilde.integrate(B3*eq_w*b3*v1*Glambda),
             T6_tilde.integrate(B3*eq_w*b3*v2*Glambda),
             T6_tilde.integrate(B3*eq_w*b3*v3*Glambda),
             T6_tilde.integrate(B3*eq_w*b3*v4*Glambda)]
>    # result (using P_2ell)
>    T6_tilde.integrate(B3*ell*b3*v2*Glambda)
<    [0, 0, 0, 0]
```

```
<    2

>    # 10M surface
>    # roots (if comparing with 1109.3454, note that e.g.
             # P_4 = P_14 + P_24 + P_4D ~ 4th root)
>    [T6_tilde.integrate(B3*eq_w*b5*v1*Glambda),
             T6_tilde.integrate(B3*eq_w*b5*v2*Glambda),
             T6_tilde.integrate(B3*eq_w*b5*v3*Glambda),
             T6_tilde.integrate(B3*eq_w*b5*v4*Glambda)]
>    # result (using the component w/ P_4D)
>    T6_tilde.integrate(B3*(v2+v3+ell+x)*b5*v4*Glambda)
<    [0, 0, 0, 0]
<    -4

>    # (non-split) 5H surface
>    # roots
>    [T6_tilde.integrate(B3*eq_w*(b3+b4)*v1*Glambda),
             T6_tilde.integrate(B3*eq_w*(b3+b4)*v2*Glambda),
             T6_tilde.integrate(B3*eq_w*(b3+b4)*v3*Glambda),
             T6_tilde.integrate(B3*eq_w*(b3+b4)*v4*Glambda)]
>    # result (using P_3H)
>    T6_tilde.integrate(B3*(b5+y)*(b3+b4)*v3*Glambda)
<    [0, 0, 0, 0]
<    2

>    # 5Hd
>    # roots
>    [T6_tilde.integrate(eq_w*(z2*z4)*v1*Glambda),
             T6_tilde.integrate(eq_w*(z2*z4)*v2*Glambda),
             T6_tilde.integrate(eq_w*(z2*z4)*v3*Glambda),
             T6_tilde.integrate(eq_w*(z2*z4)*v4*Glambda)]
>    # result
>    T6_tilde.integrate((b5+y)*(z2*z4)*v3*Glambda)
>    # for 5Hu = 5H-5Hd
>    T6_tilde.integrate((B3*(b3+b4) - z2*z4) * ((b5+y) * v3) * Glambda)
>    # ---------- CASE WHERE C_Hd = {z1,z3} ----------
>    T6_tilde.integrate((b5+y)*(z1*z3)*v3*Glambda) # 5Hd
<    [0, 0, 0, 0]
<    0
<    2
<    2
```

We can do the same for G_4^X flux, double-checking our calculations of the χ_X chiral indices in subsection D.3.1. We will leave out the roots, but they are also all zero, of course.

```
>    # 5M
>    T6_tilde.integrate(B3*ell*b3*v2*GX)
>    # 10M
>    T6_tilde.integrate(B3*(v2+v3+ell+x)*b5*v4*GX)
>    # 1
>    T6_tilde.integrate(B3*b2*b3*ell*GX)
>    # 5Hd
>    T6_tilde.integrate((b5+y)*(z2*z4)*v3*GX)
<    -9
<    9
<    -1095
<    -2
```

Appendix E

Lists

E.1 List of Figures

E.2 List of Tables

Appendix F

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27. Januar 2016 ....